

MacLaurin, Colin,

A treatise of fluxions in two books

Bd.: 2

Edinburgh (1742)

4 Math.p. 218-2

urn:nbn:de:bvb:12-bsb10053651-1



40

Math P.

218/2

Maclaurin

†

<36605749880017



<36605749880017

Bayer. Staatsbibliothek

~~In 5166~~

Mathesis. Analysis finitor. et infinitor. 346.

R

A
TREATISE
OF
FLUXIONS.

In Two BOOKS.

BY
COLIN MACLAURIN, A. M.

*Professor of Mathematics in the University of
Edinburgh, and Fellow of the Royal Society.*

VOLUME II.

EDINBURGH:
Printed by T. W. and T. RUDDIMANS.
M DCC XLII.

A
THEATRE
OF
FLUXIONS
In Two Books.

Bayerische
Staatsbibliothek
München

Colin MacLaurin, A.M.

Professor of Mathematics in the University of
Edinburgh, and Fellow of the Royal Society.

VOLUME II.

EDINBURGH.
Printed by T. W. and T. RUDIMAN.
MDCCLII.

B O O K II.

Of the Computations in the Method of Fluxions.

C H A P. I.

Of the Fluxions of Quantities considered abstractly, or as represented by general Characters in Algebra.

697. **T**HE idea of a fluxion, as described in art. 10 and 11. after Sir ISAAC NEWTON, seemed to be more immediately applicable to geometrical magnitudes, which we may very naturally conceive to be generated by motion, than to quantities considered abstractly, or as they are expressed by general symbols in algebra. For this reason we chiefly considered the fluxions of geometrical magnitudes in the first book; and most commonly gave demonstrations from geometry, because these are often preferred, as more satisfactory than algebraic computations. The evidence of the method had been disputed, and objections had been made to the number of symbols employed in it, as if they might serve to cover defects in the principles and demonstrations. In order to obviate any suspicions of this kind, we endeavoured to describe it in a manner that might represent the theorems plainly and fully, without any particular signs or characters, that they might be subjected more easily to a fair examination.

698. But an important part of this doctrine still remains to be described. The improvements that have been made by it, either in geometry or in philosophy, are in great measure owing to the facility, conciseness, and great extent of the method of computation, or algebraic part. It is for the sake of these advantages

advantages that so many symbols are employed in algebra, the number and complication of which (together with the greater care there has been taken in treating of geometry, after the excellent models left us by the ancients,) have contributed more to occasion the preference that is often ascribed to geometry, in respect of perspicuity and evidence, than any essential difference that can be supposed to be between them. It is a general kind of arithmetic; and this is what renders its usefulness so universal; nor can this be supposed to derogate from its evidence, if we have no ideas more clear or distinct than those of numbers, and often acquire more satisfactory and distinct knowledge from computations than from constructions. It may have been employed to cover, under a complication of symbols, abstruse doctrines, that could not bear the light so well in a plain geometrical form; but, without doubt, obscurity may be avoided in this art as well as in geometry, by defining clearly the import and use of the symbols, and proceeding with care afterwards.

699. The use of the negative sign in algebra is attended with several consequences that at first sight are admitted with difficulty, and has sometimes given occasion to notions that seem to have no real foundation. It implies that the real value of the quantity represented by the letter to which it is prefixed is to be subtracted; and it serves, with the positive sign, to keep in view what elements or parts enter into the composition of quantities, and in what manner, whether as increments or decrements, (that is, whether by addition or subtraction) which is of the greatest use in this art. In consequence of this, it serves to express a quantity of an opposite quality to the positive, as a line in a contrary position, a motion with an opposite direction, or a centrifugal force in opposition to gravity; and thus often saves the trouble of distinguishing, and demonstrating separately, the various cases of propositions, and preserves their analogy in view. But as the proportion of lines depends on their magnitude only, without regard to their position; and motions and forces are said to be equal, or unequal in any given ratio, without regard to their directions; and in general the proportion of quantities relates to their magnitude only, without determining whether they are to be considered as increments

ments or decrements; so there is no ground to imagine any other proportion of $-b$ and $+a$ (or of -1 and 1) than that of the real magnitudes of the quantities represented by b and a , whether these quantities are in any particular case to be added or subtracted. It is the same thing to subtract a decrement as to add an equal increment, or to subtract $-b$ from $a - b$ as to add $+b$ to it: and because multiplying a quantity by a negative number implies only a repeated subtraction of it, the multiplying $-b$ by $-n$, is subtracting $-b$ as often as there are units in n ; and is therefore equivalent to adding $+b$ so many times, or the same as adding $+nb$. But if we infer from this, that 1 is to $-n$ as $-b$ to nb , according to the rule that unit is to one of the factors as the other factor is to the product; there is no ground to imagine that there is any mystery in this, or any other meaning than that the real magnitudes represented by 1 , n , b and nb are proportional. For that rule relates only to the magnitude of the factors and product, without determining whether any factor, or the product, is to be added or subtracted. But this likewise must be determined in algebraic computations; and this is the proper use of the rules concerning the signs, without which the operation could not proceed. Because a quantity to be subtracted is never produced, in composition, by any repeated addition of a positive, or repeated subtraction of a negative, a negative square number is never produced by composition from a root. Hence the $\sqrt{-1}$, or the square-root of a negative, implies an imaginary quantity, and in resolution is a mark or character of the impossible cases of a problem; unless it is compensated by another imaginary symbol or supposition, when the whole expression may have a real signification. Thus $1 + \sqrt{-1}$, and $1 - \sqrt{-1}$ taken separately are imaginary, but their sum is 2 ; as the conditions that separately would render the solution of a problem impossible, in some cases destroy each others effect when conjoined. In the pursuit of general conclusions, and of simple forms for representing them, expressions of this kind must sometimes arise where the imaginary symbol is compensated in a manner that is not always so obvious. By proper substitutions, however, the expression may be transformed into another,

nother, wherein each particular term may have a real signification, as well as the whole expression. — The theorems that are sometimes briefly discovered by the use of this symbol, may be demonstrated without it by the inverse operation, or some other way; and tho' such symbols are of some use in the computations in the method of fluxions, its evidence cannot be said to depend upon any arts of this kind. We have just mentioned these things without enlarging upon them, for we suppose that the common algebra is admitted.

700. The rules for the computations in this method may be deduced from art. 99. but it may be worth while to demonstrate them here briefly, from general principles that may seem more immediately applicable to algebraic quantities. Any quantities that are produced from each other by an algebraic operation, or whose relation is expressed by any algebraic form, being supposed to increase or decrease together, some will be found to increase or decrease by greater differences, or at a greater rate, others by less differences, or at a less rate; and while some are supposed to increase or decrease at one constant rate by equal successive differences, others increase or decrease by differences that are always varying. We have no occasion for considering such quantities in this doctrine as generated by motions, and for enquiring into the velocities of those motions, or for considering the prime or ultimate ratio of their increments or decrements, but for ascertaining the respective rates according to which they increase or decrease, when they are supposed to vary together; in order from these to discover the properties of the quantities themselves. Thus by comparing the velocities of points that are supposed to generate lines at the same time, it appears when a line increases at a greater or less rate than another, and in what proportion. The same is to be said of any quantities, which, while they vary together, are always in the same proportion to one another as those lines. But it does not seem to be necessary to have always recourse to such suppositions; tho' in treating of geometrical magnitudes, that are often conceived to be generated by motion, this method of comparing the rates of their increase or decrease be natural and clear, and has other advantages †. When a quantity

† Art. 474.

tity A increases by differences equal to a , $2A$ increases or decreases by differences equal to $2a$, and manifestly increases or decreases at a greater rate than A in the proportion of $2a$ to a , or 2 to 1; and if m and n be invariable, $\frac{mA}{n}$ increases or decreases by differences equal to $\frac{ma}{n}$, and therefore at a greater or less rate than A in proportion as $\frac{ma}{n}$ is greater or less than a , or m is greater or less than n . This seems to be easily conceived, without having recourse to any other considerations than the relations of the differences by which the quantities increase or decrease. In order therefore to avoid the frequent repetition of figurative expressions in this algebraic part as much as possible, we will endeavour to substitute in place of the definitions and axioms above, (art. 11 and 15.) others that are rather of a more general import, but are perfectly consistent with them, and are best explained by them; as other principles and propositions in algebra are commonly best illustrated from geometry.

701. By the *fluxions* of quantities we shall therefore now understand, *any measures of their respective rates of increase or decrease, while they vary (or flow) together.* There can be no difficulty in determining those measures when the quantities increase or decrease by successive differences that are always in the same invariable proportion to each other, as in the last article. While A by increasing becomes equal to $A + a$, or by decreasing equal to $A - a$, $2A$ becomes equal to $2A + 2a$, or to $2A - 2a$; and as $2A$ increases or decreases at a greater rate than A in the proportion of $2a$ to a ; so the fluxion of A being supposed equal to a , the fluxion of $2A$ must be equal to $2a$.

In the same manner the fluxion of $\frac{m}{n} \times A$ (or of $\frac{m}{n} \times A \mp e$,

supposing m , n and e to be invariable,) is $\frac{m}{n} \times a$; and since m may be to n in any assignable ratio, a quantity may be always assigned that shall increase or decrease at a greater or less rate than A in any proportion, or that shall have its fluxi-

on greater or less than the fluxion of A in any ratio. In such cases the ratio of the fluxions and that of the differences by which the quantities increase or decrease are the same.

702. But while A is supposed to increase at a constant rate by any equal successive differences, if B increase or decrease by differences that are always varying, B cannot be said to increase or decrease at any one constant rate; and it is not so obvious how, the fluxion of A being supposed equal to its increment a , the variable fluxion of B is to be determined. It cannot be supposed that the fluxions and differences are always in the same proportion in this case; but it is evident, however, that if B increase by differences that are always greater than the equal successive differences by which $\frac{m}{n} \times A$ increases, it cannot increase at a less rate than $\frac{m}{n} \times A$; and that it cannot increase at a greater rate than $\frac{m}{n} \times A$, while its successive differences are always less than those of $\frac{m}{n} \times A$. The fluxion of A being still represented by a , the fluxion of B therefore cannot be less than $\frac{m}{n} \times a$ in the former case, or greater than $\frac{m}{n} \times a$ in the latter. The following propositions are consequences of this, and will enable us to determine at what rate B increases when its relation to A is known.

703. The successive values of the root A being represented by $A - a$, A , $A + a$, &c. which increase by any constant difference a , let the corresponding values of any quantity produced from A by any algebraic operation (or that has a dependence upon it so as to vary with it) be $B - b$, B , $B + b$, &c. Then if the successive differences b , b , &c. of the latter quantity always increase, how small soever a may be, then B cannot be said to increase at so great a rate as a quantity that increases uniformly by equal successive differences greater than b , or at so small a rate as any quantity that increases uniformly by equal successive differences less than b . In like manner, if the relation of the quantities is such, that the successive differences

rences, b , b , &c. continually decrease; then B cannot be said to increase at the same rate as a quantity that increases uniformly by equal successive differences greater than b , or less than b .

704. Therefore the fluxion of A being supposed equal to the increment a , the fluxion of B cannot be greater than b or less than b , when the successive differences b , b , &c. continually increase; and cannot be greater than b or less than b , when these successive differences always decrease.

705. In the same manner if the latter quantity decrease while the former increases, and its successive values be $B + b$, B , $B - b$, &c. then if the decrements b , b , &c. continually increase, B cannot be said to decrease at so great a rate as a quantity that decreases uniformly by equal successive differences greater than b , or at so low a rate as a quantity that decreases uniformly by equal successive differences less than b . Therefore, in this case the fluxion of A being supposed equal to a , the fluxion of B cannot be greater than b or less than b . And in the same manner if the successive decrements b , b , &c. always decrease, the fluxion of B cannot be greater than b or less than b .

706. As the fluxions of quantities are any measures of the respective rates according to which they increase or decrease, by art. 701. so it is of no importance how great or small soever those measures are, if they be in the just proportion or relation to each other. Therefore if the fluxions of A and B may be supposed equal to a and b , respectively, they may be likewise supposed equal to $\frac{1}{2}a$ and $\frac{1}{2}b$, or to $\frac{ma}{n}$ and $\frac{mb}{n}$. These principles (as other algebraic propositions) may be illustrated from geometry, as we observed in art. 700. And the propositions concerning the fluxions of areas, ordinates, &c. in the first book may be demonstrated immediately from them; but this would be needless.

707. PROP. I. *The fluxion of the root A being supposed equal to a , the fluxion of the square AA will be equal to $2A \times a$.*

Let the successive values of the root be $A - u$, A , $A + u$, and the corresponding values of the square will be $AA - 2Au$

$2Au$

712. PROP.

$2Au + uu$, AA , $AA + 2Au + uu$, which increase by the differences $2Au - uu$, $2Au + uu$, &c. and because those differences increase, it follows from art. 704. that if the fluxion of A be represented by u , the fluxion of AA cannot be represented by a quantity that is greater than $2Au + uu$, or less than $2Au - uu$. This being premised, suppose, as in the proposition, that the fluxion of A is equal to a ; and if the fluxion of AA be not equal to $2Aa$, let it first be greater than $2Aa$ in any ratio, as that of $2A + o$ to $2A$, and consequently equal to $2Aa + oa$. Suppose now that u is any increment of A less than o ; and because a is to u as $2Aa + oa$ to $2Au + ou$, it follows (art. 706.) that if the fluxion of A should be represented by u , the fluxion of AA would be represented by $2Au + ou$, which is greater than $2Au + uu$. But it was shewn from art. 704. that if the fluxion of A be represented by u the fluxion of AA cannot be represented by a quantity greater than $2Au + uu$. And these being contradictory, it follows that the fluxion of A being equal to a , the fluxion of AA cannot be greater than $2Aa$. If it can be less than $2Aa$, when the fluxion of A is supposed equal to a , let it be less in any ratio of $2A - o$ to $2A$, and therefore equal to $2Aa - oa$. Then because a is to u as $2Aa - oa$ is to $2Au - ou$, which is less than $2Au - uu$, (u being supposed less than o , as before,) it follows that if the fluxion of A was represented by u , the fluxion of AA would be represented by a quantity less than $2Au - uu$, against what has been shewn from art. 704. Therefore the fluxion of A being supposed equal to a , the fluxion of AA must be equal to $2Aa$.

708. The fluxions of A and B being supposed equal to a and b , respectively, the fluxion of $A + B$ will be $a + b$, the fluxion of $\overline{A + B^2}$, or of $AA + 2AB + BB$, will be $2 \times \overline{A + B} \times \overline{a + b}$ or $2Aa + 2Bb + 2Ba + 2Ab$, by the last article. The fluxion of $AA + BB$ is $2Aa + 2Bb$, by the same; consequently the fluxion of $2AB$ is $2Ba + 2Ab$; and the fluxion of AB is $Ba + Ab$. Hence if P be equal to AB , and the fluxion of P be p , then p will be equal to $Ba + Ab$, and dividing by P , or AB , we find $\frac{p}{P} = \frac{a}{A} + \frac{b}{B}$. If $Q = \frac{A}{B}$, and q be the fluxion of Q , then

then $QB = A$, $\frac{q}{Q} + \frac{b}{B} = \frac{a}{A}$ or $\frac{q}{Q} = \frac{a}{A} - \frac{b}{B}$; and consequently $q = \frac{Qa}{A} - \frac{Qb}{B} = \frac{a}{B} - \frac{Ab}{BB}$ or $\frac{aB - Ab}{BB}$. When any of the quantities decreases its fluxion is to be considered as negative.

709. If n be any integer number, and the sum of the terms E^{n-1} , $E^{n-2}F$, $E^{n-3}F^2$, $E^{n-4}F^3$, &c. continued till their number be equal to n , be multiplied by $E - F$, the product will be $E^n - F^n$. For the terms being formed by subducting continually unit from the index of E and adding it to the index of F , the last term will be F^{n-1} . The product of their sum multiplied by E will be $E^n + E^{n-1}F + E^{n-2}F^2 + \dots + EF^{n-1}$; their sum multiplied by $-F$ gives $-E^{n-1}F - E^{n-2}F^2 - \dots - EF^{n-1} - F^n$; and the sum of these two products is $E^n - F^n$.

710. Supposing E to be greater than F , $E^n - F^n$ will be less than $nE^{n-1} \times E - F$, but greater than $nF^{n-1} \times E - F$. For each of the terms E^{n-1} , $E^{n-2}F$, $E^{n-3}F^2$, &c. is greater than the subsequent term in the same ratio that E is greater than F , and E^{n-1} is the greatest term; consequently the number of terms being equal to n , nE^{n-1} is greater than their sum; and $nE^{n-1} \times E - F$ is greater than their sum multiplied by $E - F$, or (by the last article) greater than $E^n - F^n$. Because the last term F^{n-1} is less than any preceeding term, $nF^{n-1} \times E - F$ is less than the sum of the terms multiplied by $E - F$, or less than $E^n - F^n$.

711. When n is any integer positive number, the root A being supposed to increase by any equal successive differences, the successive differences of the power A^n will continually increase. For let $A - a$, A , $A + a$, be any successive values of the root, and $\overline{A - a}^n$, A^n , $\overline{A + a}^n$ will be the corresponding values of the power. But $\overline{A + a}^n - A^n$ is greater than $nA^{n-1}a$; as appears by substituting in the last article, $A + a$ for E , A for F , and a for $E - F$. In like manner $nA^{n-1}a$ is greater than $A^n - \overline{A - a}^n$. Therefore $\overline{A + a}^n - A^n$ is greater than $A^n - \overline{A - a}^n$, and the successive differences of the power continually increase.

712. PROP.

712. PROP. II. *The fluxion of the root A being supposed equal to a, the fluxion of the power A^n will be naA^{n-1} .*

For if the fluxion of A can be greater than naA^{n-1} , let the excess be equal to any quantity r ; suppose o equal to the excess of $\sqrt[n-1]{A^{n-1} + \frac{r}{na}}$ above A , and consequently $\overline{A+o}^{n-1} = A^{n-1} + \frac{r}{na}$. Then $na \times \overline{A+o}^{n-1}$ will be equal to $naA^{n-1} + r$, the fluxion of A^n . Let u be any increment of A less than o ; and because a is to u as $na \times \overline{A+o}^{n-1}$ to $nu \times \overline{A+o}^{n-1}$, it follows (by art. 706.) that if the fluxion of A be now represented by the increment u , the fluxion of A^n will be represented by $nu \times \overline{A+o}^{n-1}$ which is greater than $nu \times \overline{A+u}^{n-1}$, and this last is itself greater than $\overline{A+u}^n - A^n$, by art. 710. But when the successive values of the root are $A-u$, A , $A+u$, those of the power are $\overline{A-u}^n$, A^n , $\overline{A+u}^n$, the successive differences of which continually increase; consequently (by art. 704.) if the fluxion of A be represented by u , the fluxion of A^n cannot be represented by a quantity greater than $\overline{A+u}^n - A^n$, or less than $A^n - \overline{A-u}^n$. And these being contradictory, it follows that when the fluxion of A is supposed equal to a , the fluxion of A^n cannot be greater than naA^{n-1} . If it can be less than naA^{n-1} , let it be equal to $naA^{n-1} - r$, or (by supposing $o = A - \sqrt[n-1]{A^{n-1} - \frac{r}{na}}$) to $na \times \overline{A-o}^{n-1}$. Then u being supposed less than o , if the fluxion of A was represented by u , the fluxion of A^n would be represented by $nu \times \overline{A-o}^{n-1}$, which is less than $nu \times \overline{A-u}^{n-1}$ (because we suppose u to be less than o) and therefore less than $A^n - \overline{A-u}^n$ by art. 710. But this is repugnant to what has been demonstrated from art. 704. Therefore the fluxion of A being supposed equal to a , the fluxion of A^n must be equal to naA^{n-1} .

713. The

713. The fluxions of $\frac{1}{A^n}$, or of $A^{-\frac{1}{n}}$, may be determined in the same manner: but these being comprehended in the following theorem, it is needless to consider them separately. We shall only observe that the *lemma* for determining the former is, that

when E is greater than F, $\frac{1}{F^n} - \frac{1}{E^n}$ or $\frac{E^n - F^n}{E^n F^n}$ is less than

$\frac{nE^{n-1}}{E^n F^n} \times \overline{E-F}$ (by art. 710.) or $\frac{nE - nF}{EF^n} \times \overline{E-F}$ which is

less than $\frac{nE - nF}{F^n + 1}$; but greater than $\frac{nF^{n-1}}{E^n F^n} \times \overline{E-F}$ (art. 710.)

and consequently greater than $\frac{nE - nF}{E^n + 1}$. And hence it may

be demonstrated, as in art. 712. that when the fluxion of A is

supposed equal to a , the fluxion of $\frac{1}{A^n}$ is $\frac{-na}{A^{n+1}}$, the sign being

negative because $\frac{1}{A^n}$ decreases while A increases. We have

supposed n to be an integer positive number in this and the last article.

714. PROP. III. The fluxion of A being supposed equal to a , the

fluxion of $A^{\frac{m}{n}}$ will be $\frac{ma}{n} \times A^{\frac{m}{n}-1}$.

First, Let the exponent $\frac{m}{n}$ be any positive fraction whatsoever;

suppose $A^{\frac{m}{n}} = K$; consequently $A = K^{\frac{n}{m}}$; and the fluxion of K being supposed equal to k , $maA^{m-1} = nkK^{n-1}$, by art. 712.
E e e e and

and k or the fluxion of $A^{\frac{m}{n}}$ will be equal to $\frac{maA^{m-1}}{nK^{n-1}} =$

$\frac{maK}{nA} = \frac{m}{n} \times aA^{\frac{m}{n}-1}$. When $\frac{m}{n}$ is negative, let it be equal to

$-\frac{r}{n}$; and suppose $A^{-r} = K$, or $1 = A^r K$; then taking the fluxions (by art. 708.) $rA^{r-1}aK + kA^r = 0$, and $k = -$

$$\frac{rA^{r-1}aK}{A} = -rA^{-r-1}a = \frac{m}{n} \times A^{\frac{m}{n}-1}a$$

715. PROP. IV. Suppose P to be the product of any factors A, B, C, D, E , &c. (or $P = ABCDE$ &c.) let the fluxions of P, A, B, C, D, E , be respectively equal to p, a, b, c, d, e , &c. and $\frac{p}{P}$ will be equal to $\frac{a}{A} + \frac{b}{B} + \frac{c}{C} + \frac{d}{D}$ &c.

Let Q be equal to the product of all the factors of P , the first A excepted; that is, suppose $P = AQ$. Suppose R equal to the product of all the factors the first two A and B excepted, that is, let $P = ABR$ or $Q = BR$. In the same manner let $R = CS$, $S = DT$, and so on. Then, the fluxions of Q, R, S, T , &c. being supposed respectively equal to q, r, s, t , &c. it follows from art. 708. that $\frac{p}{P} = \frac{a}{A} + \frac{q}{Q} =$ (because $\frac{q}{Q} = \frac{b}{B}$

$$+ \frac{r}{R}) \frac{a}{A} + \frac{b}{B} + \frac{r}{R} = (\text{because } \frac{r}{R} = \frac{c}{C} + \frac{s}{S}) \frac{a}{A} + \frac{b}{B} + \frac{c}{C} + \frac{s}{S} = (\text{because } \frac{s}{S} = \frac{d}{D} + \frac{t}{T}) \frac{a}{A} + \frac{b}{B} + \frac{c}{C} + \frac{d}{D} + \frac{t}{T},$$

and so on. Therefore $\frac{p}{P}$ is equal to the sum of the quotients, when the fluxion of each factor of P is divided by the factor itself.

716. If the factors be supposed equal to each other, and their

their number be equal to n , then $P = A^n$, and by the last proposition $\frac{p}{P} = \frac{na}{A}$; consequently $p = \frac{nPa}{A} = naA^{n-1}$; as we found in art. 712.

717. PROP. V. If $P = \frac{ABC \times \&c.}{KLM \times \&c.}$ and the fluxions of the respective quantities be expressed by the small letters, $p, a, b, c, \&c.$ as formerly, then $\frac{p}{P} = \frac{a}{A} + \frac{b}{B} + \frac{c}{C} - \frac{k}{K} - \frac{l}{L} - \frac{m}{M}, \&c.$

For $PKLM \times \&c. = ABC \times \&c.$ and, by art. 715. $\frac{p}{P} + \frac{k}{K} + \frac{l}{L} + \frac{m}{M}, \&c. = \frac{a}{A} + \frac{b}{B} + \frac{c}{C}, \&c.$ whence by transposition, $\frac{p}{P} = \frac{a}{A} + \frac{b}{B} + \frac{c}{C} - \frac{k}{K} - \frac{l}{L} - \frac{m}{M}, \&c.$

718. The fluxion of the logarithms being supposed invariable, the fluxions of any quantities N and M will be in the same proportion as these quantities themselves. For it is the fundamental property of the logarithms, that when they are taken in any arithmetical progression, the quantities of which they are the logarithms are always in a geometrical progression. Therefore, the logarithms being supposed to increase by any equal differences, these quantities will increase or decrease by differences that increase or decrease in the same proportion as the quantities themselves. Let $A = a, A, A + a$ be the respective logarithms of $N = n, N, N + n$; and $B = a, B, B + a$ the logarithms of $M = m, M, M + m$; then because the logarithms increase by the constant difference a , n will be to n as N to $N + n$; m to m as M to $M + m$; and n to m as $N + n$ to M . Therefore when the quantities and their logarithms increase together, it follows from art. 704. that if the constant fluxion of the logarithm be supposed equal to its increment a , the fluxion of N will not be greater than n , or the fluxion of M less than m ; consequently the fluxion of N is to the fluxion of N in a ratio that is not greater than that of n to m , or of $N + n$ to M . But if the fluxion of N could be to the

fluxion of M in any ratio greater than that of N to M , as in that of $N + u$ to M ; then by supposing n to be less than u , the fluxion of N would be to the fluxion of M in a ratio greater than that of $N + u$ to M . And these being contradictory, it follows that the ratio of these fluxions is not greater than that of N to M . In the same manner the fluxion of M is to the fluxion of N in a ratio that is not greater than that of M to N . Therefore the ratio of the fluxions of M and N is the same with the ratio of the quantities M and N . When the quantities decrease while the logarithms increase, the demonstration is the same.

719. PROP. VI. *The fluxion of any quantity N is to the fluxion of its logarithm as N is to the modulus of the logarithmic system.*

For the quantities and their logarithms being supposed to increase or decrease together, when the quantity increases or decreases at the same rate as its logarithm, it is then equal to the modulus. Suppose this quantity to be M , and since the fluxion of N is to the fluxion of M as N is to M , by the last article; it follows that the fluxion of N is to the fluxion of its logarithm as N is to the modulus. Hence if $N = A^e$, e being any invariable exponent, the $\log. N = e \times \log. A$, consequently, the fluxions of N and A being supposed equal to n and a respectively, $\frac{Mn}{N} = \frac{eMa}{A}$, and $n = \frac{eNa}{A} = eA^{\frac{e-1}{A}}$. We insisted on this, at some length, in chap. 6. book I.

720. When the fluxion of a quantity is variable, it may be considered as a fluent; and its fluxion may be determined (which is called the second fluxion of that quantity) by the preceding propositions. Thus we found in art. 707. that the fluxion of A being supposed equal to a , the fluxion of AA is $2Aa$; and if A be supposed to increase at an uniform rate, or its fluxion a be invariable, $2Aa$ will increase by equal successive differences; consequently its fluxion, or the second fluxion of AA , will be equal to any of those differences (art. 701.) as to $2a \times A + a - 2Aa$, or $2aa$. If a be variable, let its fluxion be equal

equal to z , and the fluxion of $2Aa$ (or second fluxion of AA) will be $2aa + 2Az$, by art. 708. In the same manner, the fluxion of A being constant, the fluxion of $nA^{n-1}a$, or the second fluxion of A^n , is $na \times \overline{n-1} \times A^{n-2}a$, or $n \times \overline{n-1} \times aa A^{n-2}$; the fluxion of this, or the third fluxion of A^n , is $n \times \overline{n-1} \times \overline{n-2} \times a^3 A^{n-3}$. And the fluxion of A^n of any order denoted by m , is $n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}$, &c. $\times a^m A^{n-m}$, where the factors in the coefficient are to be continued till their number be equal to m . When n is any integer positive number, the fluxion of A^n , of the order n , is invariable and equal to $n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}$ &c. $\times a^n$. The quantities that represent those fluxions of A^n depend on a , which represents the fluxion of A . When A remains of the same value, the first fluxion of A^n is greater or less in the same proportion as a is supposed to be greater or less; the second fluxion of A^n is in the duplicate ratio of a ; and its fluxion of the order m is as a^m . If a be variable, but z the fluxion of a , or the second fluxion of A , be constant, then the fourth fluxion of AA will be constant and equal to $6zz$; for we found that the second fluxion of AA was $2aa + 2Az$; the fluxion of which is $4az + 2az$, or $6az$; and the fluxion of this is $6zz$. In like manner the sixth fluxion of A^3 will be constant in this case and equal to $90z^3$.

721. The second Differences of any quantity B are the successive differences of its first differences; and as the fluxion of B increases when its successive differences increase, so its second fluxion, or its fluxions of any higher order increase, when its second or higher differences increase. If we arrive at differences of any order that are constant, the fluxion of the same order is constant, and is express'd by that difference. Thus when A is supposed to increase by constant differences equal to a , and its fluxion is supposed equal to a , the second difference of AA (or

(or $\overline{A+a^2} - 2AA + \overline{A-a^2}$) is $2aa$, which is likewise its second fluxion; and the third difference of A is $6a$, which is its third fluxion. When n is any integer and positive number, the fluxion of A^n of the order n is equal to the fluxion of any of its first differences of the next inferior order, or to the fluxion of any of its second differences of the order $n-2$, and so on. For the fluxion of $\overline{A+a^n} - A^n$ (one of the first differences of A^n) of the order $n-1$ is $n \times \overline{A+a^{n-1}} - A^{n-1} \times \overline{A+a^{n-1}} - A^{n-1} \times a^{n-1} = n \times \overline{A+a^{n-1}} - A^{n-1} \times a^{n-1}$, &c. $\times a^{n-1}$, where the coefficients are supposed to be continued till their number be $n-1$, so that the last must be 2. And this we found to be the fluxion of A^n of the order n , in the preceeding article. In the same manner the fluxion of $\overline{A+a^n} - 2A^n + \overline{A-a^n}$, (the second difference of A^n) of the order $n-2$, is equal to the fluxion of $\overline{A+a^n} - A^n$ of the order $n-1$; and consequently equal to the fluxion of A^n of the order n . These fluxions are invariable and equal to the last or invariable differences. But in other cases the fluxions of A^n of any order are less than its subsequent differences of the same order, but greater than the preceeding differences, as in art. 703.

722. The preceeding propositions are demonstrated briefly by finding the ultimate relation of the differences of the fluents, for this will determine their respective rates of increasing or decreasing, or the relation of their fluxions. Thus, because $\overline{A+a^n} - A^n$, the increment of A^n , is less than $na \times \overline{A+a^{n-1}}$, but greater than naA^{n-1} , by art. 710; and when a is supposed to be diminished continually till it vanish, the ultimate ratio of $na \times \overline{A+a^{n-1}}$ to naA^{n-1} is a ratio of equality: it follows that the ultimate ratio of the increment $\overline{A+a^n} - A^n$ to naA^{n-1} is a ratio of equality; and that the fluxion of A being supposed equal to a , the fluxion of A^n must be naA^{n-1}

as

as in art. 712. In the same manner the second or higher fluxions of A^n , or of any other fluent, are ultimately equal to the corresponding differences of the fluent. If we suppose (with Mr. LEIBNITZ and those who have followed his method) a to be an infinitely small difference of A , and suppose quantities to be equal when their difference is infinitely less than the quantities themselves, $na \times \overline{A+a}^{n-1}$ must be supposed equal to naA^{n-1} ; and, since $\overline{A+a}^n - A^n$, the difference of A^n , cannot be greater than the former, or less than the latter, (art. 710.) it must be supposed equal to naA^{n-1} .

C H A P. II.

Of the notation of fluxions, the rules of the direct method, and the fundamental rules of the inverse method of fluxions.

723. SIR ISAAC NEWTON, on some occasions †, represented the fluents by capital letters, and their fluxions by the small letters that correspond to them. We followed this notation in the last chapter, in demonstrating the grounds of the operations. But it is convenient that the fluxions should be distinguished from other algebraic expressions, and in such a manner that the second and higher fluxions may be represented so as to preserve the original fluent in view. In his last method he represented the variable or flowing quantities by the final letters of the alphabet, as x, y, z ; their first fluxions by the same letters pointed once, as by $\dot{x}, \dot{y}, \dot{z}$; their second fluxions by the same letters pointed twice, as by $\ddot{x}, \ddot{y}, \ddot{z}$; the third fluxions by the letters pointed thrice, as by $\dddot{x}, \dddot{y}, \dddot{z}$, and so on; where the number of points serves to shew the order of the fluxion that is represented with respect to the first fluent; and the difference of those numbers shew of what order any of them

† Princip. lib. ii. lemm. 2.

them is the fluxion of those that preceed it, as $\dot{y}$ is the first fluxion of y , but the second fluxion of y . Mr. LEIBNITZ represented the infinitely small differences of x, y, z , by dx, dy, dz ; their second differences by ddx, ddy, ddz ; and their infinitesimal differences of any order n , by $d^n x, d^n y, d^n z$. The symbol $\dot{x}$, or dx , expresses the fluxion of x generally, without determining whether it is to be considered as positive or negative, that is, whether x increases or decreases with respect to the other fluents. Invariable quantities are represented by the first letters of the alphabet, as a, b, c , &c. These have no fluxions; and in the same manner, when any fluxion is supposed constant, its fluxion vanishes. Sir ISAAC NEWTON has comprehended most of the rules of the direct method in one general proposition; but it is more usual to represent them separately, and it may be of use to proceed gradually from the simple cases to those that are more complex.

724. I. When one simple fluent only enters each term of a compound quantity, the fluxion of this quantity is found by collecting the fluxions of each term, or by placing a point over each fluent. Thus the fluxion of $x + y - z$, is $\dot{x} + \dot{y} - \dot{z}$; the fluxion of $ax + by - cz$ is $a\dot{x} + b\dot{y} - c\dot{z}$. The fluxion of ax , or of $ax + lb$, is $a\dot{x}$. This rule is obvious, and follows from art. 701. or art. 36. 41. and 78.

725. II. As the fluxion of xy is $\dot{x}y + y\dot{x}$ (by art. 708. and 99.) so the fluxion of a product of any two fluents is the sum of the several products when the fluxion of each factor is multiplied by the other factor. Thus the fluxion of $\overline{a+x} \times \overline{b-y}$ is $\dot{x} \times \overline{b-y} - \dot{y} \times \overline{a+x} = \dot{x}b - \dot{x}y - y\dot{a} - y\dot{x}$. As the fluxion of ax is $a\dot{x}$; so the fluxion of axy is $a \times \dot{x}y + y\dot{x} = a\dot{x}y + y\dot{x}a$.

726. III. As the fluxion of the fraction $\frac{x}{y}$ is $\frac{\dot{x}y - y\dot{x}}{yy}$ by art. 708. so the fluxion of any fraction is found by multiplying the fluxion of the numerator by the denominator, subtracting the product of the fluxion of the denominator multiplied by the nu-

numerator, and dividing the remainder by the square of the denominator. Thus the fluxion of $\frac{a-x}{a+x}$ is $\frac{-\dot{x} \times a + x - \dot{x} \times a - x}{a+x^2}$

$$= \frac{-2ax}{aa + 2ax + xx.}$$

727. IV. As the fluxion of $x^{\frac{m}{n}}$ is $\frac{m}{n} \times x^{\frac{m}{n}-1} \times \dot{x}$, by art.

714. and 719. so the fluxion of a power of any invariable exponent is found by multiplying by the exponent, subtracting unit in the index of the power, and multiplying by the fluxion of the root. Thus the respective fluxions of $x^2, x^3,$

x^4 , &c. are $2x^{2-1} \times \dot{x}$ or $2xx$, $3 \times x^{3-1} \times \dot{x}$ or $3x^2 \dot{x}$,

$4 \times x^{4-1} \times \dot{x}$ or $4x^3 \dot{x}$, &c. In order to give this rule its full extent, and to reduce fluxions to the most simple expressions, we are to suppose from the common algebra, that a quantity may be carried from the numerator of a fraction to its denominator, or from the denominator to the numerator, providing the sign of its index or exponent be changed. Thus the fluxions of

$\frac{1}{x}$ or x^{-1} , $\frac{1}{xx}$ or x^{-2} , $\frac{1}{x^3}$ or x^{-3} , are respectively

$-1 \times x^{-1-1} \times \dot{x}$ or $-\frac{\dot{x}}{x^2}$, $-2 \times x^{-2-1} \times \dot{x}$ or $-\frac{2\dot{x}}{x^3}$,

$-3 \times x^{-3-1} \times \dot{x}$ or $-\frac{3\dot{x}}{x^4}$; and the fluxion of $\frac{1}{x^n}$ or x^{-n} is

$-n \times x^{-n-1} \times \dot{x}$ or $-\frac{n\dot{x}}{x^{n+1}}$. The fluxions of surds are found

by expressing them as powers with fractional exponents. Thus

the fluxion of $\sqrt{x}$ or $x^{\frac{1}{2}}$ is $\frac{1}{2} \times x^{\frac{1}{2}-1} \times \dot{x} = \frac{1}{2} \times x^{-\frac{1}{2}} \times \dot{x} = \frac{\dot{x}}{2\sqrt{x}}$

$= \frac{\dot{x}}{2\sqrt{x}}$. The fluxion of $\sqrt[3]{x}$ or $x^{\frac{1}{3}}$ is $\frac{1}{3} \times x^{\frac{1}{3}-1} \times \dot{x} = \frac{\dot{x}}{3\sqrt[3]{x}}$

$$= \frac{x}{3x^{\frac{2}{3}}} = \frac{x}{3\sqrt[3]{xx}}. \text{ The fluxion of } \sqrt[n]{x} \text{ or } x^{\frac{1}{n}} \text{ is } \frac{1}{n} \times x^{\frac{1}{n}-1} \times x$$

$$= \frac{x^{\frac{1-n}{n}}}{n} = \frac{x}{n x^n}; \text{ and the fluxion of } \frac{1}{\sqrt[n]{x}} \text{ or } x^{-\frac{1}{n}} \text{ is}$$

$$-\frac{1}{n} \times x^{-\frac{1}{n}-1} = -\frac{x}{n x^{\frac{1+n}{n}}} = -\frac{x}{n x^{1+\frac{1}{n}}}. \text{ The flu-}$$

$$\text{xion of } \frac{a+x}{b+x} \text{ is } \frac{n \times a+x}{b+x} \times x; \text{ and the fluxion of } \frac{a+x}{b+x}$$

$$\text{is } \frac{m \times a+x}{b+x} \times x \times \frac{n \times b+x}{b+x} = \frac{m \times a+x}{b+x} \times \frac{n \times b+x}{b+x} \times x$$

$$\left(\text{dividing the numerator and denominator by } b+x \right)$$

$$\frac{m \times a+x}{b+x} \times \frac{n \times b+x}{b+x} = \frac{m \times a+x}{b+x} \times \frac{n \times b+x}{b+x}$$

728. V. As when $p = x \times y \times z \times u$, &c. or to this product multiplied by any invariable quantity K , it follows from art. 719, that $\frac{\dot{p}}{p} = \frac{\dot{x}}{x} + \frac{\dot{y}}{y} + \frac{\dot{z}}{z} + \frac{\dot{u}}{u}$ &c. or that $\dot{p} = \frac{p\dot{x}}{x} + \frac{p\dot{y}}{y} + \frac{p\dot{z}}{z} + \frac{p\dot{u}}{u}$ &c. So the fluxion of any product divided by the product itself is equal to the sum of the quotients, when the fluxion of each factor is divided by the factor; or the fluxion of any product is equal to the sum of the several quantities that are formed, by substituting successively in that product the fluxion of each factor in place of the factor itself. Thus if $p = xyz$,

xyz , then $\dot{p} = \dot{x}yz + \dot{y}xz + \dot{z}xy$. If $p = \overline{a+x} \times \overline{b+x} \times \overline{c+x}$, then $\frac{\dot{p}}{p} = \frac{\dot{x}}{a+x} + \frac{\dot{x}}{b+x} + \frac{\dot{x}}{c+x}$. If $p = \overline{a+x}^m$, then $\frac{\dot{p}}{p} = \frac{mx}{a+x}$. If $p = \overline{a+x}^m \times \overline{b+x}^n \times \overline{c+x}^r$, then $\frac{\dot{p}}{p} = \frac{mx}{a+x} + \frac{nx}{b+x} + \frac{rx}{c+x}$. If $p = x + \sqrt{xx+1}$, then $\frac{\dot{p}}{p} = \frac{\dot{x} + \frac{x\dot{x}}{\sqrt{xx+1}}}{x + \sqrt{xx+1}} = \frac{\dot{x}\sqrt{xx+1} + x\dot{x}}{x + \sqrt{xx+1}} \times \frac{1}{\sqrt{xx+1}} = \frac{\dot{x}}{\sqrt{xx+1}}$. Hence if $p = \sqrt{pp+1}$, then $\frac{\dot{p}}{\sqrt{pp+1}} = \frac{mx}{\sqrt{xx+1}}$.

729. VI. As when $p = \frac{x \times y \times z}{s \times u \times t}$ &c. it follows from art. 717, that $\frac{\dot{p}}{p} = \frac{\dot{x}}{x} + \frac{\dot{y}}{y} + \frac{\dot{z}}{z} - \frac{\dot{s}}{s} - \frac{\dot{u}}{u} - \frac{\dot{t}}{t}$ &c. or $\dot{p} = \frac{px}{x} + \frac{py}{y} + \frac{pz}{z} - \frac{ps}{s} - \frac{pu}{u} - \frac{pt}{t}$ &c. So when any fraction is proposed, if we divide the fluxion of each factor of the numerator by the factor itself, and from the sum of the quotients subtract the several quotients that arise by dividing the fluxion of each factor of the denominator by this factor itself, the remainder will be equal to the fluxion of the fraction divided by the fraction; or the remainder multiplied by the fraction will give its fluxion.

Thus if $p = \frac{a+x}{a-x} \times \frac{b+x}{b-x} \times \frac{c+x}{c-x}$; then $\frac{\dot{p}}{p} = \frac{\dot{x}}{a+x} + \frac{\dot{x}}{a-x} + \frac{\dot{x}}{b+x} + \frac{\dot{x}}{b-x} + \frac{\dot{x}}{c+x} + \frac{\dot{x}}{c-x} = \frac{2a\dot{x}}{aa-xx} + \frac{2b\dot{x}}{bb-xx} + \frac{2c\dot{x}}{cc-xx}$.

730. VII. Any equation of this form $\overline{x-r} \times \overline{x-s} \times \overline{x-u} \times$ &c. = 0 being proposed, the equation for the fluxions will be $\frac{\dot{x}}{x-r} \times \frac{\dot{x}}{x-s} \times \frac{\dot{x}}{x-u} \times$ &c. = 0. For since x must be equal to r , or to s , or to u , &c. $\dot{x}$ must be equal to $\dot{r}$, or to $\dot{s}$, or to $\dot{u}$, &c.

731. VIII. Let L represent the logarithm of x , the modulus be-

being equal to a , then as $\dot{L} = \frac{ax}{x}$ by art. 721. so the fluxion of the logarithm of any Quantity is found by dividing its fluxion by the quantity itself, and multiplying by the *modulus*.

If $p = x^n$, the fluxion of the logarithm of p is $\frac{ap}{p}$ or $\frac{nax}{x}$.

The fluxion of the logarithm of $x^n y^m$ is $\frac{nax}{x} + \frac{may}{y}$. If $p = \overline{a+x} \times \overline{b+y} \times \overline{c+z} \times \&c.$ then the fluxion of the logarithm of p is $\frac{ap}{p}$ or (by art. 728.) $\frac{ax}{a+x} + \frac{ay}{b+y} + \frac{az}{c+z} + \&c.$ This likewise follows from the property of logarithms, that the logarithm of the product is equal to the Sum of the logarithms of the factors; and consequently the fluxion of the logarithm of p equal to the sum of the fluxions of the logarithms of the factors $a+x$, $b+y$, $c+z$, that is to $\frac{ax}{a+x} + \frac{ay}{b+y} + \frac{az}{c+z}$.

In the same manner the fluxion of the logarithm of $\frac{x-a}{x+a}$ is the difference of the fluxions of the logarithms of $x-a$ and $x+a$, and therefore equal to $\frac{ax}{x-a} - \frac{ax}{x+a} = \frac{2aax}{xx-aa}$. The fluxion of the logarithm of $\frac{x-a}{x+a} \times \frac{y-b}{y+b} \times \&c.$ is $\frac{2aax}{xx-aa} + \frac{2abx}{yy-bb} \&c.$

732. IX. A quantity that has a variable exponent, as y^x , is called an *exponential* or *percurrent* * quantity; and its fluxion is

$\frac{x}{a} \times y^x \times \log. y + xy^{x-1} \dot{x}$. For if we suppose $y^x = u$, then by the properties of logarithms (art. 157.) $x \times \log. y = \log. u$. And finding the fluxions by art. 725. and 731. $\dot{x} \times \log. y + \frac{ay}{y} \times x = \frac{au}{u}$; consequently the fluxion of y^x , or $\dot{u} = \frac{ux}{a} \times \log. y + \frac{y \times u}{y} = \frac{x}{a} \times y^x \times \log. y + xy^{x-1} \dot{x}$. In like manner the

flu-

*Acta Lips. 1694.

fluxions are found of exponential quantities of higher degrees.

733. X. The second fluxion is determined from the first fluxion, and the fluxion of any order from that of the preceeding order, by the same rules. It is often useful to suppose one of the variable quantities to flow uniformly, or its fluxion to be constant; in which case that quantity will have no second or higher fluxion, and the second or higher fluxions of quantities that depend upon it will be expressed in a more simple manner. Thus the fluxion of x being supposed constant, the first fluxion of x^n being $n\dot{x}x^{n-1}$, its second fluxion will be $n \times \overline{n-1} \times \dot{x}^2 x^{n-2}$, its third fluxion $n \times \overline{n-1} \times \overline{n-2} \times \dot{x}^3 x^{n-3}$; and its fluxion of any order m will be $n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \&c. \times \dot{x}^m x^{n-m}$, where the factors in the coefficient are to be continued till their Number be equal to m .

734. The second or higher fluxions of quantities may be found (without computing those of the preceeding orders) by particular theorems, as in the last example. Thus the fluxion of xy is $\dot{x}y + y\dot{x}$; the second fluxion of xy is therefore $\ddot{x}y + 2\dot{x}\dot{y} + x\ddot{y}$; its third fluxion is $\dddot{x}y + 3\ddot{x}\dot{y} + 3\dot{x}\ddot{y} + x\dddot{y}$; and in general the fluxion of xy of any order denoted by m is found by multiplying the fluxion of x of the order m by y , the fluxion of x of the order $m-1$ by $\dot{y}$, the fluxion of x of the order $m-2$ by $\ddot{y}$, and proceeding always in this manner (diminishing the order of the fluxion of x and increasing the order of the fluxion of y by unit) then prefixing to the several products the respective coefficients of the binomial $1+1$ raised to the power m ; the last term being the product of x by the fluxion of y of the order m . If we suppose the fluxion of x to be constant, then the two last terms will give the fluxion of xy of the order required: And if the second fluxion of x be constant, the three last terms will give that fluxion of xy ; and so on. When the fluxion of x of any order r , and the fluxion of y of any order s , are supposed constant, the fluxion of xy of any order m , (supposing m not to exceed $r+s$) is determined by this theorem.

735. In the inverse method, it is required to find the fluent when

when the fluxion is given; and the rules are derived from those of the direct method; as the rules in division and evolution in algebra are deduced from those of multiplication and involution. As when a fluent consists of a variable and an invariable part, the latter does not appear in the fluxion; so when any fluxion is proposed, it is only the variable part of the fluent that can be derived from it. If $\dot{x}$ represent any fluxion that may be proposed, the variable part of the fluent will be equal to x ; for supposing y to be any variable quantity, if $x + y$ could represent the fluent of $\dot{x}$, then $\dot{x} + \dot{y}$ would be equal to $\dot{x}$, and $\dot{y} = 0$, or y would be invariable, against the supposition. But supposing K to represent any invariable quantity, then $x + K$ may generally represent the fluent of $\dot{x}$. If it be required to find such a fluent of $\dot{x}$ as shall vanish when x is supposed to vanish, it can be no other than x ; and if it be required that the fluent should vanish when x is equal to any given quantity a , then by supposing $x + K$ to vanish when x becomes equal to a , we have $a + K = 0$, or $K = -a$; whence the fluent is $x - a$. In the same manner the fluent of $-\dot{x}$ may be generally represented by $K - x$. When a fluxion, that is proposed, coincides with any of those which were deduced from their fluents in any of the preceeding articles, the variable part of the fluent required must coincide with that which was there proposed. As division in algebra leads us to fractions, and evolution to surds; so the inverse method of fluxions leads us often to quantities that are not known in the common algebra, and that cannot be expressed by the common algebraic symbols. In the following articles we will endeavour to give some account of the Progress that has been made in this method.

736. I. As the fluxion of $ax + by - cz$ is $a\dot{x} + b\dot{y} - c\dot{z}$; so conversely, when any aggregate of quantities is proposed, each of which involves a simple fluxion that is not multiplied by any flowing quantity, the variable part of the fluent is found by substituting in place of each fluxion its particular fluent; or by taking away the points, or other fluxionary symbols. Thus the variable part of the fluent of $a\dot{x} + b\dot{y} - c\dot{z}$ is $ax + by - cz$. If it is required that this fluent should vanish when x vanishes, let y be

y be then equal to e , and z equal to f ; and the fluent will be $ax + b \times \overline{y-e} - c \times \overline{z-f}$. For the whole fluent may be expressed by $ax + by - cz + K$, where K is supposed invariable. But, by the supposition when x vanishes, this fluent vanishes, and is equal to $be - cf + K$; whence $K = -be + cf$; and consequently $ax + by - cz + K$ is equal to $ax + by - be - cz + cf$ or to $ax + b \times \overline{y-e} - c \times \overline{z-f}$.

737. II. As the fluxion of x^n is $nx^{n-1} \dot{x}$ by art. 727. so conversely, when the fluxion proposed is the product of any power of a variable quantity multiplied by its fluxion, with any invariable coefficient, the variable part of the fluent is found by adding unit to the exponent of the Power, dividing by the exponent thus increased and by the fluxion of the root. Thus the variable part of the fluent of $nx^{n-1} \dot{x}$ is $\frac{nx^{n-1+1} \dot{x}}{n-1+1 \times x} = x^n$; and if it is required that the fluent should vanish when x vanishes, it is then precisely x^n ; but if it is to vanish when x is equal to any given quantity a , the whole fluent is $x^n - a^n$. In general we may express it by $x^n + K$, where K may represent any invariable quantity. In the same manner the fluent of $axx \dot{x}$ is $a \times \frac{x^{1+1} \dot{x}}{2x} + K = \frac{1}{2} axx + K$; the fluent of $ax^2 \dot{x}$ is $\frac{ax^{2+1} \dot{x}}{3x} + K = \frac{1}{3} ax^3 + K$; the fluent of $\frac{ax}{x^2}$, or $ax^{-2} \dot{x}$, is $\frac{ax^{-2+1} \dot{x}}{-1 \times x} + K = -\frac{a}{x} + K$; the fluent of $x^{\frac{1}{2}} \dot{x}$ is $\frac{2x^{\frac{1}{2}+1} \dot{x}}{3x} + K = \frac{2x^{\frac{3}{2}}}{3} + K$. The fluent of an aggregate of quantities of this kind is found by computing the fluent of each term separately. Thus the fluent of $x^2 \dot{x} + axx \dot{x} + bbx \dot{x}$ is $\frac{1}{3} x^3 + \frac{1}{2} ax^2 + bbx + K$; the fluent of $xx \times \overline{a+x^2}$, or of $axxx \dot{x} + 2ax^2 \dot{x} + x^3 \dot{x}$ is $\frac{1}{2} a^2 x^2 + \frac{2ax^3}{3} + \frac{x^4}{4}$. The fluent of $x^m \dot{x} \times \overline{x+a}$ when

when n is an integer, is found by raising $x \mp a$ to the power n , multiplying each term by $x^m \dot{x}$, finding the fluent of each separately by this rule, and collecting them into one sum. The variable part of the fluent is assignable in all those cases, unless when the fluxion $\frac{\dot{x}}{x}$ or $\dot{x}x^{-1}$ is involved in one of the terms; of which case we are to treat afterwards.

738. III. As the fluxion of xy is $\dot{x}y + y\dot{x}$, by art. 725, so when any proposed fluxion can be resolved into two terms of this form, where there are two fluxions, each of which is separately multiplied by the fluent of the other, then the product of the two fluents is the variable part of the fluent required. Thus the fluent of $b\dot{z} - u\dot{z} - a\dot{u} - z\dot{u}$, or $\dot{z} \times b - u - \dot{u} \times a + z$ is $b - u \times a + z + K$. In the same manner when a fluxion can be resolved into three parts in the form $\dot{x}yz + y\dot{x}z + z\dot{x}y$, where there are three fluxions x, y, z , and each of these is separately multiplied by the product of the fluents of the other two fluxions, then xyz the product of the three fluents is the variable part of the fluent required. These theorems are easily continued from art. 728.

739. IV. The fluxion of $e\dot{x}z + \dot{z}x$, where e is supposed to be invariable, is not of the same form with any of those in the preceeding article, but by multiplying it by x^{e-1} , the product $e\dot{x}x^{e-1}z + \dot{z}x^e$ is easily reduced to the first of them. For supposing $y = x^e$, $e\dot{x}x^{e-1}z + \dot{z}x^e = y\dot{z} + \dot{z}y$, the fluent of which is yz , or zx^e ; which is therefore the fluent of $e\dot{x}z + \dot{z}x \times x^{e-1}$. In the same manner if $e\dot{x}yz + f\dot{y}xz + z\dot{x}y$ be multiplied by $x^{e-1}y^{f-1}$, the fluent of the product will be $x^e y^f z$. And when the fluxion $e\dot{x}yzu + f\dot{y}xzu + g\dot{z}xyu + u\dot{x}yz$ is multiplied by $x^{e-1}y^{f-1}z^{g-1}$, the fluent is $x^e y^f z^g u$; and so on. It follows from the first of these, that when an equation $e\dot{x}z + \dot{z}x = ax^m \dot{x}$ is proposed the equation for

for the fluents is $zx^e + K = \frac{1}{m+1} \times ax^{m+1}$; and in this manner the fluents in art. 540. were found.

740. V. When $\frac{\dot{p}}{p} = \frac{\dot{x}}{x} + \frac{\dot{y}}{y} + \frac{\dot{z}}{z} + \&c.$ then we may conclude that p is the product of $x, y, z, \&c.$ and of some invariable quantity K ; for this fluxional equation will arise (by art. 728.) when we suppose $p = Kxyz \times \&c.$ If $\frac{\dot{p}}{p} = \frac{\dot{x}}{x} + \frac{\dot{y}}{y} + \frac{\dot{z}}{z} + \frac{\dot{s}}{s} + \frac{\dot{u}}{u} + \&c.$ then we may conclude that $p = \frac{Kxyz}{su} \times \&c.$ Thus if $\frac{\dot{p}}{p} = \frac{\dot{x}}{x+a} - \frac{\dot{x}}{x+b}$, we may conclude that $p = K \times \frac{x+a}{x+b}$. If $\frac{ep}{p} = \frac{mx}{x} + \frac{ny}{y}$, (e, m and n being supposed invariable) then $p^e = K x^m y^n$.

741. VI. A fluxion that is not proposed under any of the preceeding forms may in some cases, by a proper substitution, be changed into an equal fluxion that will appear under one or more of them; and thus the fluent may be discovered. The fluxion $\dot{z}^n \times \overline{a+z}^m$ is not immediately comprehended under any of the preceeding forms when m is a fraction, or any negative number. But by supposing $x = a + z$, or $z = x - a$, and consequently $\dot{z} = \dot{x}$, and $\overline{z}^n = \overline{x-a}^n$, the proposed fluxion is transformed into $\dot{x}^m \times \overline{x-a}^n$; the fluent of which is found by raising $x-a$ to the power of the exponent n , multiplying each term by $x^m \dot{x}$, and computing the fluent of each product separately by art. 737.

742. The fluxion $\dot{x}^m \times \overline{e+fx}^n$ being proposed; suppose $e + fx^n = z$, and $\frac{m+1}{n} = r$; then $\overline{x}^n = \frac{z-e}{f}$, $\overline{x}^{m+1} = \frac{z-e}{f}^{\frac{m+1}{n}}$, and (by taking the fluxions) $\frac{m+1}{n} \times \frac{\dot{z}}{f} = \frac{\dot{z}}{fr}$, and

G g g g

$\overline{x}^m \dot{x}$

$x^m \dot{x} = \frac{r}{f^r} \times \overline{z-e}^{r-1} \times \dot{z}$. Therefore the fluxion that was
 proposed will be equal to $\frac{r}{m+1 \times f^r} \times \overline{z-e}^{r-1} \times \dot{z} \dot{z} =$
 $\frac{\dot{z} \dot{z}}{n f^r} \times \overline{z-e}^{r-1}$; consequently the fluent is found by rai-
 sing $z-e$ to the power of the exponent $r-1$, multiplying each
 term of this power by $\dot{z} \dot{z}$, finding the fluent of each product
 separately by art. 737. and dividing the sum of these fluents by
 $n f^r$. This fluent is assignable in finite terms when r or $\frac{m+1}{n}$
 is an integer, (unless l be of such a value as to give occasion to
 the exception mentioned above at the end of art. 737.) and will
 consist of as many terms as there are units in r ; because this is
 the number of terms in the power of $z-e$ of the exponent $r-1$.
 For example the fluent of $x^m \dot{x} \times \overline{e+fx^l}$ is assignable in alge-
 braic terms equal in number to $m+1$, when m is any integer
 and positive number; for in this case $n=1$ and $r=m+1$.
 The fluent of $x^m \dot{x} \times \overline{e+fx^2}^l$ is assignable in finite terms when
 m is any odd positive number; because in this case $n=2$, and
 $r = \frac{m+1}{2} = \frac{m+1}{2}$ which is an integer when m is an odd posi-
 tive number. The fluxion $x^m \dot{x} \sqrt{e+fx^2} = x^{m+\frac{1}{2}} \dot{x} \times$
 $\overline{e+fx^2}^{\frac{1}{2}}$; and consequently the fluent is assignable when $m+\frac{3}{2}$
 is an integer positive number, that is when m is equal to any
 fraction of this series $-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \&c.$ The fluent of $x^s \dot{x}$
 $\times \overline{e+fx^{\frac{k}{s}}}$ is assignable in finite terms when $s+1$ is any mul-
 tiple of k ; for in this case r (or $\frac{1}{s}+1$ divided by $\frac{k}{s}$) is equal
 to $\frac{1+s}{k}$, and is an integer when $s+1$ is a multiple of k .

743. The

743. The same fluxion $\dot{x}^m \dot{x} \times e + f x^{nl}$ (multiplying the first part $\dot{x}^m \dot{x}$ by x^{nl} , and dividing the other part $e + f x^{nl}$ by the same x^{nl} , by which their product is not altered,) is expressed by $\dot{x} x^{m+nl} \times \frac{e x^{-n} + f}{-n}$. As the value of r taken from the first expression was $\frac{m+1}{n}$, so its value computed from the second expression is $\frac{m+nl+1}{-n}$. Therefore the fluent is assignable in a finite number of algebraic terms, not only when $\frac{m+1}{n}$ is an integer and positive number, but likewise when $\frac{m+nl+1}{-n}$ is such a number. Thus the fluent of $\dot{x} x e + f x^{nl}$ is assignable in a finite number of terms when k is integer and positive, whatever number be represented by n ; for in this case $m = 0, l = -k - \frac{1}{n} = \frac{-nk-1}{n}$, $nl + 1 = -nk$, and $\frac{m+nl+1}{-n} = \frac{-nk}{-n} = k$.

744. When the difference or sum of two fluents is invariable, their fluxions are equal, as we observed in art. 735. And hence when the same fluxion is represented by two different expressions, as in the two preceeding articles, there may be some difference betwixt the fluents that are derived from them by the preceeding rules; but by the addition or subtraction of an invariable quantity, they will be found to agree with one another.

Thus, for example, the fluent of $\dot{x} \times a + x^{-2}$ is (by art. 737.)

$$\frac{\dot{x}}{x} \times \frac{1}{a+x} = \frac{1}{a+x}. \text{ The same fluxion is equal to } x^{-2} \dot{x} \times$$

$$a x^{-1} + 1^{-2}, \text{ and the fluent of this fluxion (by the same ar-}$$

$$\text{ticle) is } \frac{x^{-2} \dot{x} \times a x^{-1} + 1^{-1}}{-a x^{-2} \dot{x}} = \frac{a x^{-1} + 1}{-a} = \frac{-1}{a \times a x^{-1} + 1}$$

$$= \frac{-x}{a \times a + x}. \text{ The latter fluent vanishes when } x \text{ vanishes. The}$$

former $\frac{1}{a+x}$, by adding the invariable quantity K , becomes

$$G g g g 2$$

$$K +$$

$K + \frac{1}{a+x}$; and if we suppose this fluent to vanish when x vanishes, $K + \frac{1}{a+0} = 0$, $K = -\frac{1}{a}$, and the fluent will be $-\frac{1}{a} + \frac{1}{a+x} = \frac{-a-x+a}{a \times a+x} = \frac{-x}{a \times a+x}$, which coincides with the latter fluent.

745. VII. When a fluent cannot be represented accurately in algebraic terms, it is then to be expressed by a converging series, or by a more simple fluent that is already known. In division in the common algebra (and in decimal arithmetic) the quotient is often such a series. Let $\frac{ax}{a-x}$ be the fluxion proposed; and if we divide a by $a-x$ by the usual method, we shall find the quotient or $\frac{a}{a-x} = 1 + \frac{x}{a} + \frac{x^2}{a^2} + \frac{x^3}{a^3} \&c.$ Hence $\frac{ax}{a-x} = \dot{x} + \frac{xx}{a} + \frac{x^2 \dot{x}}{a^2} + \frac{x^3 \dot{x}}{a^3} \&c.$ and the fluent of $\frac{ax}{a-x}$ is equal (by finding the fluents of the terms $\dot{x}$, $\frac{xx}{a}$, $\frac{x^2 \dot{x}}{a^2}$, $\&c.$ separately from art. 737.) to the series $x + \frac{x^2}{2a} + \frac{x^3}{3a^2} + \frac{x^4}{4a^3} + \&c.$ which may be of use for determining the fluent when x is very small in respect of a ; because, in that case, a few terms at the beginning of the series will be nearly equal to the value of the whole. This series gives us the logarithm of $\frac{aa}{a-x}$, the *modulus* being supposed equal to a , by art. 731. For if we suppose $\frac{aa}{a-x} = z$, then $\frac{\dot{x}}{a-x} = \frac{\dot{z}}{z}$, by art. 728. and the fluent of $\frac{ax}{a-x}$ is equal to $\log. z$ or $\log. \frac{aa}{a-x}$, or to $-\log. a-x$.

746. In the same manner $\frac{aa}{aa+xx} = 1 - \frac{x^2}{a^2} + \frac{x^4}{a^4} - \frac{x^6}{a^6} \&c.$ and the fluent of $\frac{aax}{aa+xx}$ is the fluent of $\dot{x} - \frac{x^2 \dot{x}}{a^2} + \frac{x^4 \dot{x}}{a^4} - \frac{x^6 \dot{x}}{a^6} \&c.$

$-\frac{x^6 \dot{x}}{a^6}$ &c. that is (by art. 737.) $x - \frac{x^3}{3a^2} + \frac{x^5}{5a^4} - \frac{x^7}{7a^6}$ &c.

Because the fluxion of the arch is to the fluxion of its tangent in the duplicate ratio of the radius to the secant (by art. 195.) it follows that if the radius be a , the tangent x , and consequently the secant equal to $\sqrt{aa+xx}$, the fluxion of the arch will be equal to $\frac{aax}{aa+xx}$; and the ark itself will be expressed by the se-

ries $x - \frac{x^3}{3a^2} + \frac{x^5}{5a^4} - \frac{x^7}{7a^6}$ &c. or $x \times 1 - \frac{x^2}{3a^2} + \frac{x^4}{5a^4} - \frac{x^6}{7a^6}$ &c.

This series was given by Mr. JAMES GREGORY for computing the arch from its tangent. *Commer. epistol.* 1671. Dr. HALLEY has computed the ratio of the circumference of the circle to its diameter from it, by supposing x to be the tangent of an arch of 30 gr. in which case the tangent x is to the secant $\sqrt{aa+xx}$ as 1 to 2, and consequently x to a as 1 to $\sqrt{3}$; so that the arch

of 30 gr. is the product of $\frac{a}{\sqrt{3}}$ multiplied by the series $1 -$

$\frac{1}{9} + \frac{1}{45} - \frac{1}{189} + \frac{1}{729}$ &c. and the whole circumference to the diameter as $\sqrt{12}$ multiplied by this series to unit. This series may

be represented by $1 - \frac{1}{3 \times 3} + \frac{1}{5 \times 9} - \frac{1}{7 \times 27} + \frac{1}{9 \times 81} -$

$\frac{1}{11 \times 243}$ &c. that the law of its continuation may appear.

747. In like manner, when the roots of powers are extracted by the usual rules in Algebra, the root is often expressed by a

series of this kind. Thus $\sqrt{aa-xx} = a - \frac{x^2}{2a} - \frac{x^4}{8a^3} -$

$\frac{x^6}{16a^5}$ &c. consequently $x \sqrt{aa-xx} = ax - \frac{x^3}{2a} - \frac{x^5}{8a^3} -$

$\frac{x^7}{16a^5}$ &c. Therefore the fluent of $x \sqrt{aa-xx}$ is (by art. 737.)

$ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7}$ &c. And if CA the radius of

the circle be represented by a , upon which CP be taken from the center C equal to x , CB and PM perpendicular to CA meet the
circle

circle in B and M; then the Area CBMP will be express'd by this series; for $PM = aa - xx$, the fluxion of the area CBMP (art. 107.) equal to $PM \times \dot{x} = \dot{x} \sqrt{aa - xx}$; and consequently the area CBMP equal to the fluent. Let MN be perpendicular to CB in N, and the area $BMN = CBMP - CP \times PM = CBMP - x \sqrt{aa - xx} = ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} \&c.$

$$- ax + \frac{x^3}{2a} + \frac{x^5}{8a^3} + \frac{x^7}{16a^5} \&c. = \frac{x^3}{3a} + \frac{x^5}{10a^3} + \frac{3x^7}{56a^5} \&c.$$

748. Because the fluxion of the arch BM is to $\dot{x}$ the fluxion of its sine MN or CP, as CB to PM, that is as a to $\sqrt{aa - xx}$, the fluxion of BM is expressed by $\frac{a\dot{x}}{\sqrt{aa - xx}} = \frac{a\dot{x}\sqrt{aa - xx}}{aa - xx} =$

(dividing the series which expresses $\sqrt{aa - xx}$ by $aa - xx$) $\dot{x} + \frac{x^2\dot{x}}{2a^2} + \frac{3x^4\dot{x}}{8a^4} + \frac{5x^6\dot{x}}{16a^6} + \&c.$ consequently the arch BM is equal to $x + \frac{x^3}{6a^2} + \frac{3x^5}{40a^4} + \frac{5x^7}{112a^6} \&c. = x + \frac{1 \times 1}{2 \times 3} \times \frac{x^2 A}{a^2} +$

$\frac{3 \times 3}{4 \times 5} \times \frac{x^2 B}{a^2} + \frac{5 \times 5}{6 \times 7} \times \frac{x^2 C}{a^2} + \&c.$ where A represents the first term x , B the second term $\frac{x^2 A}{6a^2}$, C the third term, and so on.

It is useful to represent a series in this manner, that it may be easily continued to any number of terms, and the fluent computed to any degree of exactness that may be required. Let the arch NS described from the center C meet CM in S, and NS will be to BM as CN to CB, that is as $\sqrt{aa - xx}$ to a ; consequently, if the series which expresses BM be multiplied by

the series which expresses $\frac{\sqrt{aa - xx}}{a}$, viz. $1 - \frac{x^2}{2a^2} - \frac{x^4}{8a^4} - \frac{x^6}{16a^6} \&c.$ the product $x - \frac{x^3}{3a^2} - \frac{2x^5}{15a^4} - \frac{8x^7}{105a^6} \&c.$ will

represent the ark NS. Therefore $MN - NS = \frac{x^3}{3a^2} + \frac{2x^5}{15a^4}$

$+ \frac{8x^7}{105a^6} \&c.$ And the area BMN is to $CB \times \overline{MN - NS}$ as

$$\frac{x^3}{3a}$$

$\frac{x^3}{3a} + \frac{x^5}{10a^3} + \frac{3x^7}{56a^5}$ &c. to $\frac{x^3}{3a^2} + \frac{2x^5}{15a^4} + \frac{8x^7}{105a^6}$ &c. or as
 $1 + \frac{3x^2}{10a^2} + \frac{9x^4}{56a^4}$ &c. to $1 + \frac{2x^2}{5a^2} + \frac{8x^4}{35a^4}$ &c. This ratio
 by substituting $\frac{c}{b}$ instead of $\frac{x}{a}$ coincides with that which was given
 in art. 655. without a Proof, as the ratio (FIG. 294) of the seg-
 ment FCO to $CD \times \overline{CF-CS}$.

748. Sir ISAAC NEWTON's binomial theorem is of excellent
 use for extracting the roots of powers, or reducing a quantity to
 a series of this kind; and, having made no use of this theorem in
 demonstrating the rules in the direct method of fluxions, we
 may the rather give an investigation of it from art. 727. Let it
 be required to find $\overline{1+x^n}$, where n may represent any integer,
 number or fraction, whether it be positive or negative. It is
 evident from what is shown in the common algebra concerning
 powers and their roots, that the first term of any power of
 $1+x$ is 1, and that the subsequent terms involve $x, x^2, x^3, x^4,$
 &c. with invariable coefficients. Suppose therefore $\overline{1+x^n} =$
 $1 + Ax + Bx^2 + Cx^3 + Dx^4 + \&c.$ where $A, B, C, D, \&c.$
 represent any such coefficients. By finding the fluxions (art.
 727.) $n\dot{x} \times \overline{1+x^{n-1}} = A\dot{x} + 2Bx\dot{x} + 3Cx^2\dot{x} + 4Dx^3\dot{x}$
 $+ \&c.$ and dividing by $n\dot{x}$, we have $\overline{1+x^{n-1}} = \frac{A}{n} + \frac{2Bx}{n}$
 $+ \frac{3Cx^2}{n} + \frac{4Dx^3}{n} + \&c.$ And since this equation must be
 true, whatever the value of x may be, it follows by supposing
 $x = 0$, (or because the first term of $\overline{1+x^{n-1}}$ must be 1) that $\frac{A}{n}$
 $= 1$. and $A = n$. By taking the fluxion of the last equation,
 $\overline{n-1} \times \overline{1+x^{n-2}} \times \dot{x} = \frac{2B\dot{x}}{n} + \frac{6Cx^2\dot{x}}{n} + \frac{12Dx^3\dot{x}}{n} + \&c.$
 and dividing by $\overline{n-1} \times \dot{x}$, we have $\overline{1+x^{n-2}} = \frac{2B}{n \times n-1} +$
 $\frac{6Cx^2}{n \times n-1}$

$\frac{6Cx}{n \times n-1} + \frac{12Dx^2}{n \times n-1} + \&c.$ and by supposing $x = 0$, (or because the first term of any power of $1 + x$ must be 1) $\frac{2B}{n \times n-1} = 1$

or $B = n \times \frac{n-1}{2}$. By taking the fluxions again we find $\frac{1}{1+x^{n-3}} \times \dot{x} = \frac{6Cx}{n \times n-1} + \frac{24Dxx}{n \times n-1} \&c.$ and $1 + x^{n-3} =$

$\frac{6C}{n \times n-1 \times n-2} + \frac{24Dx}{n \times n-1 \times n-2} + \&c.$ so that $\frac{6C}{n \times n-1 \times n-2} =$

1, or $C = n \times \frac{n-1}{2} \times \frac{n-2}{3}$; and so on. Therefore $1+x^n =$

$1 + nx + n \times \frac{n-1}{2} \times x^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times x^3 + \&c.$

And $a+b = \frac{a+b^n}{a^n} \times a^n = 1 + \frac{b}{a} \Big|^n \times a^n =$ (by substitut-

ing $\frac{b}{a}$ for x) $a^n + \frac{na^n b}{a} + n \times \frac{n-1}{2} \times \frac{a^n b^2}{a^2} + n \times \frac{n-1}{2}$

$\times \frac{n-2}{3} \times \frac{a^n b^3}{a^3} + \&c. = a^n + na^{n-1}b + n \times \frac{n-1}{2} \times a^{n-2}b^2$

$+ n \times \frac{n-1}{2} \times \frac{n-2}{3} \times a^{n-3}b^3 + \&c.$ which is the binomial theorem.

749. In the same manner if we suppose $a + bx + cxx + dxx^2 \&c. = A + Bx + Cx^2 + Dx^3 \&c.$ by supposing $x = 0$, we have

$A = a^n$. By taking the fluxions, and dividing by $\dot{x}$, we shall find

$\frac{a + bx + cxx \&c.}{a + bx + cxx \&c.} \times nb + 2ncx + 3ndx \&c. = B + 2Cx + 3Dx^2 + \&c.$ and by supposing $x = 0$, we have $B =$

$na^{n-1}b$. By taking the fluxions again, dividing by $2\dot{x}$, and then

supposing $x = 0$, we shall find $C = n \times \frac{n-1}{2} \times a^{n-2}bb +$

$na^{n-1}c$. And by proceeding in the same manner, we may in-

vestigate the other coefficients D, E, &c. in Mr. DE MOIVRE'S

theorem for raising a multinomial to any power of the index n .

Of

Of this the reader will find a fuller account in the *Philosoph. Transf. n. 230.* or *Miscel. Analyt. p. 87.* There are several other methods by which these theorems are investigated; but we have described that which is immediately suggested by the method of fluxions, and will be of use afterwards in other enquiries.

750. When any fluxion $\dot{x}P$ is proposed, and P is any quantity that can be expressed by any Powers of x and invariable quantities, the value of P can be resolved into a series by these theorems; and each term being multiplied by $\dot{x}$, the fluent of each may be found separately by art. 737, such excepted as are of the same form with $A\dot{x}x^{-1}$. Thus to find the fluent of $x^m \dot{x} \times$

$\overbrace{e + fx^n}^l$, it is first transformed by supposing $z = e + fx^n$ into $\frac{zz^l}{nf^r} \times \overbrace{z - e}^{r-1}$ (as in art. 742.) = (by the binomial theorem) $\frac{zz^l}{nf^r} \times z^{r-1} - \frac{r-1}{1} \times z^{r-2} e + \frac{r-2}{2} \times z^{r-3} ee - \&c.$

consequently the fluent is the product of $\frac{1}{nf^r}$ multiplied by $\frac{z^{l+r}}{l+r}$ $-\frac{r-1}{l+r-1} \times z^{l+r-1} e + \frac{r-1}{2} \times \frac{r-2}{l+r-2} \times z^{l+r-2} ee \&c.$

which (by restoring $e + fx^n$ for z , supposing $l+r = s$, and $rn - n = p$) will be found equal to $\frac{1}{nsf^r} \times \overbrace{e + fx^n}^{l+1} x^p -$

$\frac{r-1}{s-1} \times \frac{eA}{fx^n} + \frac{r-2}{s-2} \times \frac{eB}{fx^n} - \frac{r-3}{s-3} \times \frac{eC}{fx^n} \&c.$ where A represents the first term, B the second, C the third, and so on. This series was given long ago by Sir ISAAC NEWTON, *Commer. epistol.* And in his Treatise of quadratures, he has shewn how to assign the fluent of $\dot{x}P$ in a series, when P is the product of $a + bx^n + cx^{2n} \&c.$ multiplied by x^m and by any multinomial $e + fx^n + gx^{2n} + hx^{3n} \&c.$ raised to a power of any exponent l ; or when P is equal to the product of those quantities

H h h h

ties

ties multiplied by $k + 1x^n + mx^{2n}$ &c. raised to a power of any exponent k . *De quadrat. curvar.* prop. 5. & 6.

751. The following theorem is likewise of great use in this doctrine. Suppose that y is any quantity that can be expressed by a series of this form $A + Bz + Cz^2 + Dz^3 + \&c.$ where $A, B, C, \&c.$ represent invariable coefficients as usual, any of which may be supposed to vanish. When z vanishes, let E be the value of y , and let $\dot{E}, \ddot{E}, \ddot{\ddot{E}}, \&c.$ be then the respective values of $\dot{y}, \ddot{y}, \ddot{\ddot{y}}, \&c.$ z being supposed to flow uniformly.

$$\text{Then } y = E + \frac{\dot{E}z}{z} + \frac{\ddot{E}z^2}{1 \times 2 z^2} + \frac{\ddot{\ddot{E}}z^3}{1 \times 2 \times 3 z^3} + \frac{\ddot{\ddot{\ddot{E}}}z^4}{1 \times 2 \times 3 \times 4 z^4} +$$

&c. the law of the continuation of which series is manifest. For since $y = A + Bz + Cz^2 + Dz^3 + \&c.$ it follows that when $z = 0$, A is equal to y ; but (by the supposition) E is then equal to y ; consequently $A = E$. By taking the fluxions, and

dividing by $\dot{z}$, $\frac{\dot{y}}{\dot{z}} = B + 2Cz + 3Dz^2 + \&c.$ and when

$z = 0$, B is equal to $\frac{\dot{y}}{\dot{z}}$, that is to $\dot{E}$. By taking the fluxions a-

gain, and dividing by $\dot{z}$, (which is supposed invariable) $\frac{\ddot{y}}{\dot{z}} =$

$2C + 6Dz + \&c.$ let $z = 0$, and substituting $\ddot{E}$ for $\ddot{y}$, $\frac{\ddot{E}}{\dot{z}} =$

$2C$, or $C = \frac{\ddot{E}}{2\dot{z}}$. By taking the fluxions again, and dividing by

$\dot{z}$, $\frac{\ddot{\ddot{y}}}{\dot{z}} = 6D + \&c.$ and by supposing $z = 0$, we have $D = \frac{\ddot{\ddot{E}}}{6\dot{z}}$

Thus it appears that $y = A + Bz + Cz^2 + Dz^3 + \&c. =$

$$E + \frac{\dot{E}z}{z} + \frac{\ddot{E}z^2}{1 \times 2 z^2} + \frac{\ddot{\ddot{E}}z^3}{1 \times 2 \times 3 z^3} + \frac{\ddot{\ddot{\ddot{E}}}z^4}{1 \times 2 \times 3 \times 4 z^4} + \&c.$$

This proposition may be likewise deduced from the binomial theorem.

Let

Let BD, the ordinate of the figure FDM at B, be equal to E, BP = z , PM = y , and this series will serve for resolving the value of PM, or y , (some particular cases being excepted, as when any of the coefficients $E, \frac{\dot{E}}{z}, \frac{\ddot{E}}{z^2}, \&c.$ become in-

FIG. 299.

finite) into a series, not only in such cases as were described in the preceeding articles, but likewise when the relation of y and z is determined by an affected equation, and in many cases when their relation is determined by a fluxional equation. This theorem was given by Dr. TAYLOR, *method. increm.* By supposing the fluxion of z to be represented by BP, or $z = z$, we

have $y = E + \dot{E}z + \frac{\ddot{E}z^2}{2} + \frac{\ddot{\ddot{E}}z^3}{6} + \frac{\ddot{\ddot{\ddot{E}}}z^4}{24} + \&c.$ (as was observed in art. 255.) and hence it appears at what rate the fluxion

of y of each order contributes to produce the increment or decrement of y , since $y - E = \dot{E}z + \frac{\ddot{E}z^2}{2} + \frac{\ddot{\ddot{E}}z^3}{6} + \frac{\ddot{\ddot{\ddot{E}}}z^4}{24} + \&c.$ If

Bp be taken on the other side of B equal to BP, then $pm = A - Bz + Cz^2 - Dz^3 + \&c.$ = (the same quantities being represented by $\frac{\dot{E}}{z}, \frac{\ddot{E}}{z^2}, \&c.$ as before, or the base being supposed

to flow the same way,) $E - \frac{\dot{E}z}{z} + \frac{\ddot{E}z^2}{1 \times 2z^2} - \frac{\ddot{\ddot{E}}z^3}{1 \times 2 \times 3z^3} + \frac{\ddot{\ddot{\ddot{E}}}z^4}{1 \times 2 \times 3 \times 4z^4} - \&c.$ consequently $PM + pm = 2E + \frac{2\ddot{E}z^2}{1 \times 2z^2} + \frac{2\ddot{\ddot{E}}z^4}{1 \times 2 \times 3 \times 4z^4} + \&c.$

752. The area BDMP, or the fluent of yz , is equal to the

fluent of $Ez + \frac{Ez\dot{z}}{z} + \frac{\dot{E}z^2\dot{z}}{1 \times 2z^2} + \frac{\ddot{E}z^3\dot{z}}{1 \times 2 \times 3z^3} + \&c.$ that is

(because while this area is generated by the ordinate PM, the

H h h h 2

quan-

quantities $E, \frac{\dot{E}}{z}, \frac{\ddot{E}}{z^2}, \&c.$ are invariable) to $Ez + \frac{\dot{E}z^2}{1 \times 2z} + \frac{\ddot{E}z^3}{1 \times 2 \times 3z^2} + \frac{\ddot{\ddot{E}}z^4}{1 \times 2 \times 3 \times 4z^3} + \&c.$ which theorem is not materially different from Mr. BERNOUILLI's *Act. Erud. Lips.* 1694.

In the same manner the area $BDmp = Ez - \frac{\dot{E}z^2}{1 \times 2z} + \frac{\ddot{E}z^3}{1 \times 2 \times 3z^2} - \frac{\ddot{\ddot{E}}z^4}{1 \times 2 \times 3 \times 4z^3} + \&c.$ where z is supposed to be the same as in the former case.

Therefore the area $PMmp$ bounded by the ordinates PM and pm , that are at equal distances from BD , (or E)

on opposite sides, is $2Ez + \frac{2\dot{E}z}{1 \times 2 \times 3z^2} + \frac{2\ddot{\ddot{E}}z^3}{1 \times 2 \times 3 \times 4 \times 5z^4} + \&c.$ and

is equal to the rectangle contained by the base Pp (or $2z$) and

the series $E + \frac{\dot{E}z^2}{2 \times 3 \times z^2} + \frac{\ddot{\ddot{E}}z^4}{2 \times 3 \times 4 \times 5z^4} + \&c.$

753. The series for finding the number of a given logarithm may be deduced by the theorem in art. 751. Let z represent the logarithm of y , the *modulus* being represented by M ; and

since $\frac{y}{y} = \frac{z}{M}$ by art. 731. it follows that $\dot{y} = \frac{yz}{M}$, $\ddot{y} =$

$\frac{\dot{y}z}{M} = \frac{yz^2}{M^2}$, $\ddot{\ddot{y}} = \frac{\ddot{y}z^2}{M^2} = \frac{yz^3}{M^3}$, $\ddot{\ddot{\ddot{y}}} = \frac{\ddot{\ddot{y}}z^3}{M^3} = \frac{yz^4}{M^4}$, and so on.

When $z = 0$, then $y = 1$. Therefore we are to suppose $E = 1$,

$\dot{E} = \frac{z}{M}$, $\ddot{E} = \frac{z^2}{M^2}$, $\ddot{\ddot{E}} = \frac{z^3}{M^3}$, $\ddot{\ddot{\ddot{E}}} = \frac{z^4}{M^4}$; consequently, by

the theorem, $y = 1 + \frac{z}{M} + \frac{z^2}{2M^2} + \frac{z^3}{6M^3} + \frac{z^4}{24M^4} + \&c.$

The same series is found by supposing $y = 1 + Az + Bz^2 + Cz^3 + Dz^4 + \&c.$ And therefore $M\dot{y} = yz = z + Az^2 +$

$Bz^3 + Cz^4 + \&c.$ consequently by finding the fluents, $M\dot{y}$ (or M

$M + AMz + BMz^2 + CMz^3 + \&c.) = K + z + \frac{Az^2}{2} + \frac{Bz^3}{3} + \frac{Cz^4}{4} + \&c.$ and by comparing the terms of those two values of My that involve the same powers of z , $K = M$, $AM = 1$ or $A = \frac{1}{M}$, $BM = \frac{A}{2}$ or $B = \frac{A}{2M}$, $CM = \frac{B}{3}$ or $C = \frac{B}{3M}$, and so on; therefore $y = 1 + \frac{z}{M} + \frac{z^2}{2M^2} + \frac{z^3}{6M^3} + \&c.$ By supposing $z = M$, $y = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \&c. = 2, 7182818 \&c.$ The ratio of this Number to unit is that which Mr. COTES calls the *ratio modularis*, the *modulus* being always the logarithm of this ratio in any logarithmic system. See art. 175.

754. In the same manner, the series for finding the co-sine, when the arch is given, is deduced from the theorem in art. 751. Let the arch $BM = z$, and its co-sine $PM = y$, then because

$$\dot{y} : \dot{z} :: CM : CP :: \sqrt{a^2 - y^2} : a, \quad \frac{\dot{y}^2}{z^2} = \frac{a^2 - y^2}{a^2}, \quad \frac{2\dot{y}\ddot{y}}{z^2} =$$

$$-\frac{2y\ddot{y}}{a^2} \text{ or } \frac{\ddot{y}}{z^2} = \frac{-y}{a^2}, \quad \frac{\ddot{y}}{z^2} = \frac{-\dot{y}}{a^2}, \quad \frac{\ddot{y}}{z^2} = \frac{-\ddot{y}}{a^2} = \frac{y\ddot{z}^2}{a^4}, \quad \frac{\ddot{y}}{z^2} = \frac{\dot{y}\ddot{z}^2}{a^4},$$

and so on. When the arch BM vanishes, or $z = 0$, $PM = CB = a$; supposing therefore $E = a$, substitute a for y in those values of $\dot{y}$, $\ddot{y}$, $\ddot{\ddot{y}}$, $\&c.$ in order to obtain $\dot{E}$, $\ddot{E}$, $\ddot{\ddot{E}}$, $\&c.$ and $\frac{\dot{E}}{z}$

$$= \frac{a^2 - a^2}{a^2} = 0, \quad \frac{\ddot{E}}{z} = \frac{-a}{a^2} = \frac{-1}{a}, \quad \frac{\ddot{\ddot{E}}}{z^2} = 0, \quad \frac{\ddot{\ddot{\ddot{E}}}}{z^3} = \frac{1}{a^3}, \quad \&c.$$

$$\text{Therefore } y = E + \frac{\dot{E}z}{z} + \frac{\ddot{E}z^2}{2z^2} + \&c. = a - \frac{z^2}{2a} + \frac{z^4}{24a^3}$$

$$- \frac{z^6}{720a^5} + \&c. \text{ If we suppose } BM \text{ still equal to } z, \text{ but its right}$$

$$\text{fine } MN \text{ now equal to } y, \text{ the same equation } \frac{\dot{y}^2}{z^2} = \frac{a^2 - y^2}{a^2}$$

will express the relation of $\dot{y}$ to z , and the values of $\ddot{y}$, $\ddot{\ddot{y}}$, $\&c.$ will

will be the same as in the former case. But because when BM vanishes, its sine MN likewise vanishes, we are now to suppose $E = 0$, and to substitute 0 for y in the values of $y, \dot{y}, \ddot{y},$ &c. in order to obtain $\dot{E}, \ddot{E}, \ddot{\ddot{E}},$ &c. Therefore in this case

$$\frac{\dot{E}}{z} = \frac{a^2 - 0}{a^2} = 1, \dot{E} = 0, \frac{\ddot{E}}{z^2} = \frac{1}{a^2}, \ddot{E} = 0, \text{ \&c. And } y =$$

$$z - \frac{z^3}{6a^2} + \frac{z^5}{120a^4} - \frac{z^7}{5040a^6} + \text{ \&c. If } y \text{ represent the tan-}$$

gent of the ark z , then (as in art. 746.) $\frac{\dot{y}}{z} = \frac{a^2 + y^2}{a^2}$; and sup-

posing $E = 0$, (because y vanishes with z) and proceeding as be-

$$\text{fore, we shall find } y = z + \frac{z^3}{3a^2} + \frac{2z^5}{15a^4} + \frac{17z^7}{315a^6} + \text{ \&c.}$$

If y represent the secant of the ark z , then $y : z :: y \sqrt{yy - aa} : aa$, and supposing $E = a$, because the secant becomes equal to the radius when the ark vanishes, it will be found that $y =$

$$a + \frac{z^2}{2a} + \frac{5z^4}{24a^3} + \frac{61z^6}{720a^5} + \text{ \&c. In the same manner ge-}$$

neral theorems are found for the reversion of series, such as are given by Sir ISAAC NEWTON, *commerc. epist.* in his letter of October 1676, towards the end. We now proceed with our

our account of the inverse method of fluxions; but will have occasion to return to the doctrine of series afterwards, and to shew further the use of the theorems in art. 751 and 752.

C H A P. III.

Of the analogy betwixt circular arches and logarithms, and of reducing fluents to these, or to hyperbolic and elliptic arches, or to other fluents of a more simple form; when they are not assignable in finite algebraic terms.

755. **W**hen it does not appear that a fluent can be assigned in a finite number of algebraic terms, we are not therefore to have recourse immediately to an infinite series. The arches of a circle, and hyperbolic areas or logarithms, cannot be assigned in algebraic terms, but have been computed with great exactness by several methods. By these, with algebraic quantities, any segments of conic sections and the arks of a parabola are easily measured; and when a fluent can be assigned by them, this is considered as the second degree of resolution. When it does not appear that a fluent can be measured by the areas of conic sections, it may however be measured in some cases by their arks; and this may be considered as the third degree of resolution. If it does not appear that a fluent can be assigned by the arks of any conic sections (the circle included) it may however be of some use to assign the fluent by an area or ark of some other figure that is easily constructed or described; and it is often important that the proposed fluxion be reduced to a proper form, in order that the series for the fluent may not be too complex, and that it may not converge at too slow a rate.

756. The rule in art. 737. is of no use to find the fluent of $x^{-1} \dot{x}$, or $\frac{\dot{x}}{x}$; for according to that rule, the fluent is

$$\frac{\dot{x} \times x^{-1+1}}{1-1 \times x} = \frac{\dot{x} \times x^0}{0} = \text{(because } x^0 = x^{1-1} = \frac{x}{x} = 1\text{)} \frac{1}{0};$$

from which expression no computation of the fluent can be deduced,

duced, and therefore this case was excepted. By art. 731. the fluent of $\frac{x}{x}$ is equal to $\frac{\log. x}{M}$, M being the the *modulus*, and the fluent being supposed to vanish when x is equal to 1 or to the quantity whose logarithm vanishes. If we suppose $x = a + z$, then $\frac{+ax}{x} = \frac{az}{a+z}$, and the fluent will be found (as in art. 745.) $= z \frac{+z^2}{2a} + \frac{z^3}{3a^2} + \frac{z^4}{4a^3} + \frac{z^5}{5a^4}, \&c.$ Suppose $p = \frac{a+z}{a-z}$, and (art. 728.) $\frac{\dot{p}}{p} = \frac{z}{a+z} + \frac{z}{a-z}$; consequently the fluent of $\frac{Mp}{p}$ or $\log. p = 2M \times \frac{z}{a} + \frac{z^3}{3a^3} + \frac{z^5}{5a^5} + \frac{z^7}{7a^7} + \&c.$ as in art. 173. In the same manner other theorems are found for computing logarithms.

FIG. 300. 757. The fluent of $\frac{x}{x}$ is equal to AHIE the area of the equilateral hyperbola, AH being perpendicular from the vertex A and EI from any point E to the asymptote AH in H and I, supposing OH=1, and OI=x, O being the center of the figure; because the ordinate EI = $\frac{AH \times HI}{OI} = \frac{1}{x}$. Hence the area AHIE, or the sector AOE, is called the hyperbolic logarithm of OI, or EI, the *modulus* being supposed equal to AH x OH or 1; and such coincide with the logarithms in NAPIER's first tables; whereas the tabular logarithms are now equal to these multiplied by the reciprocal of the hyperbolic logarithm of 10, as was more fully explained in art. 174. If the sector OAK : OAE :: n : 1, and KL be perpendicular to the asymptote in L, then $\log. OL = n \times \log. OI$; and $OL : OH :: OI^n : OH^n$.

758. The properties of the circle and ellipse often suggest similar properties of the hyperbola; and reciprocally the properties of hyperbolic areas (which are sometimes more easily discovered because of their analogy to the properties of logarithms described in book 1. chap. 6.) are of use for discovering the analogous properties of circular and elliptic areas. The fol-

following theorem serves to shew how great this analogy is, and leads us in a brief manner to various general theorems that relate to the multiplication and division of circular sectors or arcs. FIG. 300. & 301.
 Let O be the center of the ellipse or hyperbola AEK , OA either semi-axis of the ellipse, but the semi-transverse axis in the hyperbola, av the axis perpendicular to OA , OAK a sector that is the same multiple of the sector OAB in both figures, Kk and Bb perpendicular to av in k and b ; suppose $OA = a$, $Bb = x$, and $Kk = z$, when the perpendiculars Bb , Kk are on the same side of the axis av with OA (as they always are in the hyperbola;) but $Bb = -x$ or $Kk = -z$, when Bb , or Kk , are on the other side of av in the ellipse. Then the relation of z to x will be determined by the same equation in both figures. To make this appear, let AOB , BOC , COD , DOE , &c. be any equal sectors in the hyperbola; and let AOB , BOC , COD , DOE , &c. be likewise any equal sectors in the ellipse; let Bb , Cc , Dd , Ee , &c. be perpendicular to av in b , c , d , e , &c. in each figure; join AC , BD , CE , DF , &c. intersecting the semidiameters OB , OC , OD , OE , &c. in M , N , P , Q , &c. respectively. Because the sectors AOB , BOC , COD , DOE , &c. are equal, AC , BD , CE , DF , &c. are ordinates of the respective semi-diameters OB , OC , OD , OE , &c. For the same reason OB is to OM , OC to ON , OD to OP , OE to OQ , &c. always in the same ratio of Bb to OA , in the same figure; as was shown above of the ellipse (*introd. p. 8. § 617.*) and is easily extended to the hyperbola. Let Mm , Nn , Pp , Qq , &c. be perpendicular to the diameter av in each figure in m , n , p , q , r , &c. respectively; and the ratio of Mm to Bb , of Nn to Cc , of Pp to Dd , of Qq to Ee , &c. will be always the same as that of Bb to OA . Then because AC is bisected in M , $Cc + OA = 2Mm = 2Bb \times \frac{Bb}{OA} = 2Bb \times \frac{x}{a}$: because BD is bisected in N , $Dd + Bb = 2Nn = 2Cc \times \frac{x}{a}$. In the same manner $Ee + Cc = 2Pp = 2Dd \times \frac{x}{a}$; and so on. Therefore, since in both figures $Cc = 2Bb \times \frac{x}{a} - OA$,

$Dd = 2Cc \times \frac{x}{a} - Bb$, $Ee = 2Dd \times \frac{x}{a} - Cc$, and so on; it appears that the relation of Cc to Bb , of Dd to Bb , of Ee to Bb , and in general, the relation of Kk to Bb (the sector OAK being the same multiple of OAB in both figures) will be expressed always by the same equation in the ellipse and hyperbola, the perpendiculars Bb and Kk being on the same side of the diameter av with OA . But if the perpendicular Ff (for example) stand in the ellipse on the other side of av , then $-Ff$ will be determined from Bb and OA in the ellipse by an equation of the same form with that which serves for determining $+Ff$ from Bb and OA in the hyperbola; for in this case we find in the ellipse $Dd - Ff = Qq = 2Ee \times \frac{x}{a}$, or $-Ff = 2Ee \times \frac{x}{a} - Dd$; and in

the hyperbola $+Ff = 2Ee \times \frac{x}{a} - Dd$. In the same manner in the ellipse $-Gg = -2Ff \times \frac{x}{a} - Ee$, but $+Gg = 2Ff \times \frac{x}{a} - Ee$ in the hyperbola; whence $-Ff, -Gg, &c.$ are determined in the ellipse by the same equation as $+Ff, +Gg, &c.$ in the hyperbola. And in general it appears that $\mp Kk$ or z , is always determined from $\mp Bb$, or x , and OA , or a , in both figures by the same equation.

FIG. 300. 759. In the equilateral hyperbola, let BS and KT be perpendicular to the transverse axis in S and T , VB and LK perpendicular to the asymptote meet the same axis in X and Z ; let the sector $OAK : OAB :: n : 1$, $OX = y$, Bb or $OS = x$, and Kk or $OT = z$ as before. Then by the common Property of this hyperbola, $BS^2 = OS^2 - OA^2$, that is $BS = \sqrt{xx - aa}$, and $OX (=y) = OS + SX = OS + BS = x + \sqrt{xx - aa}$; In the same manner $K'T = \sqrt{zz - aa}$, $OZ = OT + TZ = OT + TK = z + \sqrt{zz - aa}$. Because the sector $AOK : AOB :: n : 1$, it follows (art. 757.) that $OV^n : OH^n (:: OX^n : OA^n) :: OL : OH :: OZ : OA$; that is $y^n : a^n :: z + \sqrt{zz - aa} : a$,
or

or $z + \sqrt{zz-aa} = \frac{y^n}{a^{n-1}}$, because $y = x + \sqrt{xx-aa}$, $y-x^2$
 $= xx-aa$, or $yy-2xy+aa=0$; and because $z + \sqrt{zz-aa}$
 $= \frac{y^n}{a^{n-1}}$, it follows that $y^{2n} - 2a^{n-1}zy^n + a^{2n} = 0$. Hence

the relation of z to x is found in the hyperbola by comparing
the two equations $y^{2n} - 2za^{n-1}y^n + a^{2n} = 0$, and $yy - 2xy + aa = 0$, and exterminating y . Therefore (by the last Ar- FIG. 301.
ticle) if the sector OAK be to OAB in the circle as n to 1, or
the ark AK $= n \times$ AB, then the relation of $\overline{Kk}$ (the co-sine
of the ark AK) to $\overline{Bb}$ (the co-sine of AB) will be determin-
ed by supposing $\overline{Kk} = z$, $\overline{Bb} = x$, OA $= a$, and extermi-
nating y from the two equations $y^{2n} - 2za^{n-1}y^n + a^{2n} =$
 0 , and $yy - 2xy + aa = 0$; of which theorem Mr. DE MOIVRE
has made excellent use for resolving a trinomial of the form
 $y^{2n} - 2zy^n + 1$ into quadratic trinomials (*miscel. analyt. lib. I.*)
as we shall see afterwards.

760. Produce SB and TK till they meet the asymp- FIG. 300.
tote in s and t , and $Kt : OA :: Bs^n : OA^n$; that is
 $z - \sqrt{zz-aa} : a :: x - \sqrt{xx-aa}^n : a^n$; consequently

$z - \sqrt{zz-aa} = a \times \frac{x - \sqrt{xx-aa}^n}{a}$. Therefore since $z + \sqrt{zz-aa}$
 $= a \times \frac{x + \sqrt{xx-aa}^n}{a}$, it follows (by adding those equations)

that $z = \frac{a}{2} \times \frac{x + \sqrt{xx-aa}^n + x - \sqrt{xx-aa}^n}{a}$, which (by the
binomial theorem) is equal to $\frac{1}{a^{n-1}}$ multiplied by $x^n + n \times \frac{n-1}{2}$

$\times x^{n-2} \times \overline{xx-aa} + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times x^{n-4} \times \overline{xx-aa}^2$
 $+ \&c.$ Or, the radius a being supposed equal to unit, raise
 $x + 1$ to the power of the exponent n , multiply the terms ta-
ken

ken alternately, beginning with the first x^n by 1, $xx-1$, $\frac{xx-1}{2}$, $\frac{xx-1}{3}$, &c. respectively, and the sum of the products will be equal to z . Hence if Ob the sine of the ark AB be represented by u , or $uu=aa-xx$, then Kk or z will be equal to

the product of $\frac{x^n}{a^{n-1}}$ multiplied by $1-n \times \frac{n-1}{2} \times \frac{uu}{xx} + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{u^4}{x^4} - \&c.$ It is evident from what has been

shewn that $x + \sqrt{xx-aa} = a \times \frac{z + \sqrt{zz-aa}}{a}$ and $2a^{2n-1} \propto$

$= \frac{z + \sqrt{zz-aa}}{a} + \frac{z - \sqrt{zz-aa}}{a}$. Let Ok (the sine of the ark AK) = S , or $SS = aa - zz =$ (by substituting the value of z) $2a^{2n} - \frac{x + \sqrt{xx-aa}}{4a^{2n-2}} - \frac{x - \sqrt{xx-aa}}{4a^{2n-2}}$; consequently

$S = \frac{x + \sqrt{aa-xx}}{2a^{n-1}} - \frac{x - \sqrt{xx-aa}}{2a^{n-1}} \times \sqrt{-1}$ which, (by the

binomial theorem) is equal to $\frac{1}{a^{n-1}}$ multiplied by nx^{n-1}

$- n \times \frac{n-1}{2} \times \frac{n-2}{3} \times x^{n-3} u^3 + \&c.$ The series given by

Sir ISAAC NEWTON for finding the sine of the ark AK from the sine of AB , may be derived from this theorem, or from art. 751.

FIG. 300. 761. Let Ar and AR , the tangents of the hyperbola or circle at A intercepted by the semidiameters OB and OK , be represented by t and T ; and because $Bb : Ob :: OA : Ar$, we find

in the hyperbola $x = \frac{aa}{\sqrt{aa-tt}}$ and $\sqrt{xx-aa} = \frac{at}{\sqrt{aa-tt}}$, but in

the circle $x = \frac{aa}{\sqrt{aa+tt}}$ and $\sqrt{xx-aa} = \frac{at\sqrt{-1}}{\sqrt{aa+tt}}$. By substitu-

ting these values for x and $\sqrt{xx-aa}$, and similar values for z and $\sqrt{zz-aa}$ in the first equation in art. 759. $z + \sqrt{zz-aa} =$

$a \times \frac{a + \sqrt{xx - aa}}{a}$, which was shewn to be common to both figures we have in the hyperbola $\frac{a+T}{\sqrt{aa-TT}} = \frac{a+t}{\sqrt{aa-tt}}$, or (because $\frac{a+T}{aa-TT} = \frac{a+T}{a-T}$) $\frac{a+T}{a-T} = \frac{a+t}{a-t}$, and $T = a \times \frac{a+t - a-t}{a+t + a-t}$; but in the circle $\frac{a+T\sqrt{-1}}{\sqrt{aa+TT}} = \frac{a+t\sqrt{-1}}{\sqrt{aa+tt}}$ or $\frac{a+T\sqrt{-1}}{a-T\sqrt{-1}} = \frac{a+t\sqrt{-1}}{a-t\sqrt{-1}}$ and $T = a \times \frac{a+t\sqrt{-1} - a-t\sqrt{-1}}{a+t\sqrt{-1} + a-t\sqrt{-1}}$
 $=$ (by art. 748.) $a \times \frac{na^{n-1}t - n \times \frac{n-1}{2} \times \frac{n-2}{3} \times a^{n-3}t^3 + \&c.}{a^n - n \times \frac{n-1}{2} \times a^{n-2}t^2 + \&c.}$

This theorem was given by Mr. BERNOUILLI *Act. Lips.* 1712.

762. The same theorems are immediately deduced from the inverse method of fluxions, by representing circular arks as imaginary logarithms; for in this manner an analogy is preserved in the expressions of the fluents, as near as possible to that which is betwixt their fluxions, or betwixt the equations of the circle and hyperbola. The fluxion of the hyperbolic sector OAB is to the fluxion of the triangle OAr (or $\frac{1}{2}at$) as BS^2 to Ar^2 or as $OS^2 (= OA^2 + BS^2)$ to OA^2 , and consequently as OA^2 to $OA^2 - Ar^2$, that is as aa to $aa - tt$; and is expressed by

$\frac{1}{2}at \times \frac{aa}{aa - tt}$; the fluxion of OAK is in the same manner

$\frac{1}{2}a \dot{T} \times \frac{aa}{aa - TT}$. Therefore since $OAK = n \times OAB$ we have

$\frac{aa \dot{T}}{aa - TT} = \frac{naat}{aa - tt}$. By supposing $p = a \times \frac{a+t}{a-t}$, we have (art.

728.) $\frac{\dot{p}}{p} = \frac{\dot{t}}{a+t} + \frac{\dot{t}}{a-t} = \frac{2at}{aa - tt}$, and $\frac{aat}{aa - tt} = \frac{ap}{2p}$. In the

same

same manner by supposing $q = a \times \frac{a+T}{a-T}$, the fluxion $\frac{aa\dot{T}}{aa-TT}$
 $= \frac{aq}{2q}$; consequently $\frac{ap}{2p} = \frac{naq}{2q}$, $\frac{p}{p} = \frac{nq}{q}$, and (art. 738.) $p =$
 $q^n \times K$ where K is invariable, or $a \times \frac{a+T}{a-T} = a^n K \times \frac{a+T}{a-T}^n$ or
 (because T and t vanish together, and $a^n K = a$) $\frac{a+T}{a-T} =$
 $\frac{a+t}{a-t}^n$, as in the last article.

763. In the same manner the fluxion of the circular ark AB
 viz. $\frac{aat}{aa+tt}$ (art. 746.) by supposing $p = a \times \frac{a+t\sqrt{-1}}{a-t\sqrt{-1}}$ is trans-
 formed into $\frac{ap}{2p\sqrt{-1}}$; because (art. 728.) $\frac{\dot{p}}{p} = \frac{t\sqrt{-1}}{a+t\sqrt{-1}} +$
 $\frac{i\sqrt{-1}}{a-t\sqrt{-1}} = \frac{2at\sqrt{-1}}{aa+tt}$. Therefore the circular ark is equal to

the fluent of $\frac{ap}{2p\sqrt{-1}}$, and is expressed by $\frac{a}{2M\sqrt{-1}} \times \log. p. =$
 $\frac{a}{2M\sqrt{-1}} \times \log. a \times \frac{a+t\sqrt{-1}}{a-t\sqrt{-1}}$; where the value of p is imagina-

ry, and is so far compensated by the imaginary symbol $M\sqrt{-1}$,
 that the whole compound expression may be supposed to denote
 the circular ark; as such imaginary symbols compensate each
 other in the expressions of the real roots of cubic and higher
 equations. See art 699. In the same manner the fluxion of the

circular AK , or $\frac{aa\dot{T}}{aa+TT}$, by supposing $q = a \times \frac{a+T\sqrt{-1}}{a-T\sqrt{-1}}$

is transformed into $\frac{aq}{2q}$; and since $\frac{aa\dot{T}}{aa+TT} = \frac{naat}{aa+tt}$, it fol-

lows that $\frac{\dot{q}}{q} = \frac{np}{p}$, $q = p^n \times K$, or $a \times \frac{a+T\sqrt{-1}}{a-T\sqrt{-1}} = a^n K \times$

$\frac{a+t\sqrt{-1}}{a-t\sqrt{-1}}^n$, or (because T and t vanish together, and conse-

quently $a^n K = a$) $\frac{a+T\sqrt{-1}}{a-T\sqrt{-1}} = \frac{a+t\sqrt{-1}}{a-t\sqrt{-1}}^n$, as in art. 761.

763. In

Fig. 295. N. 1.

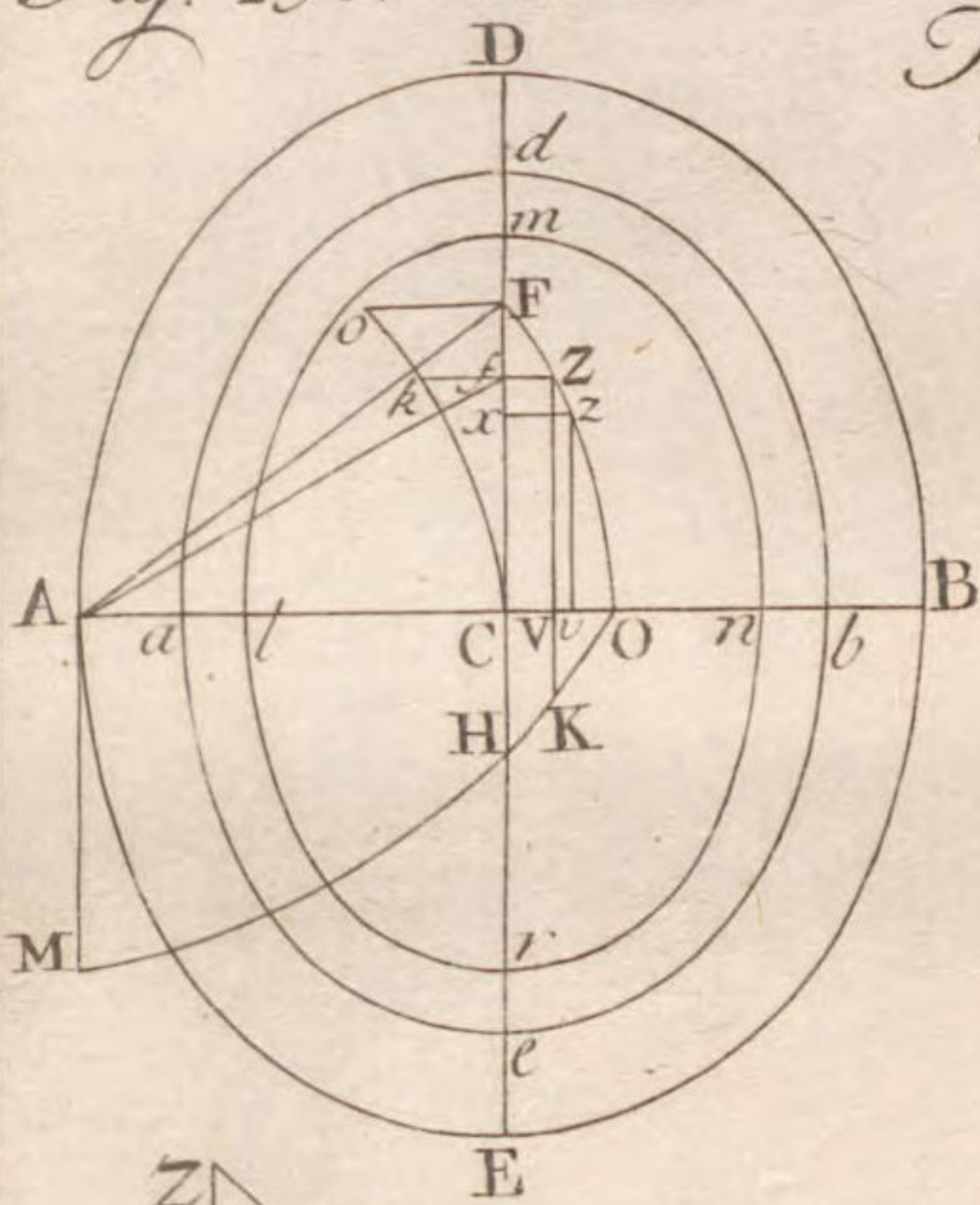


Fig. 295. N. 2.

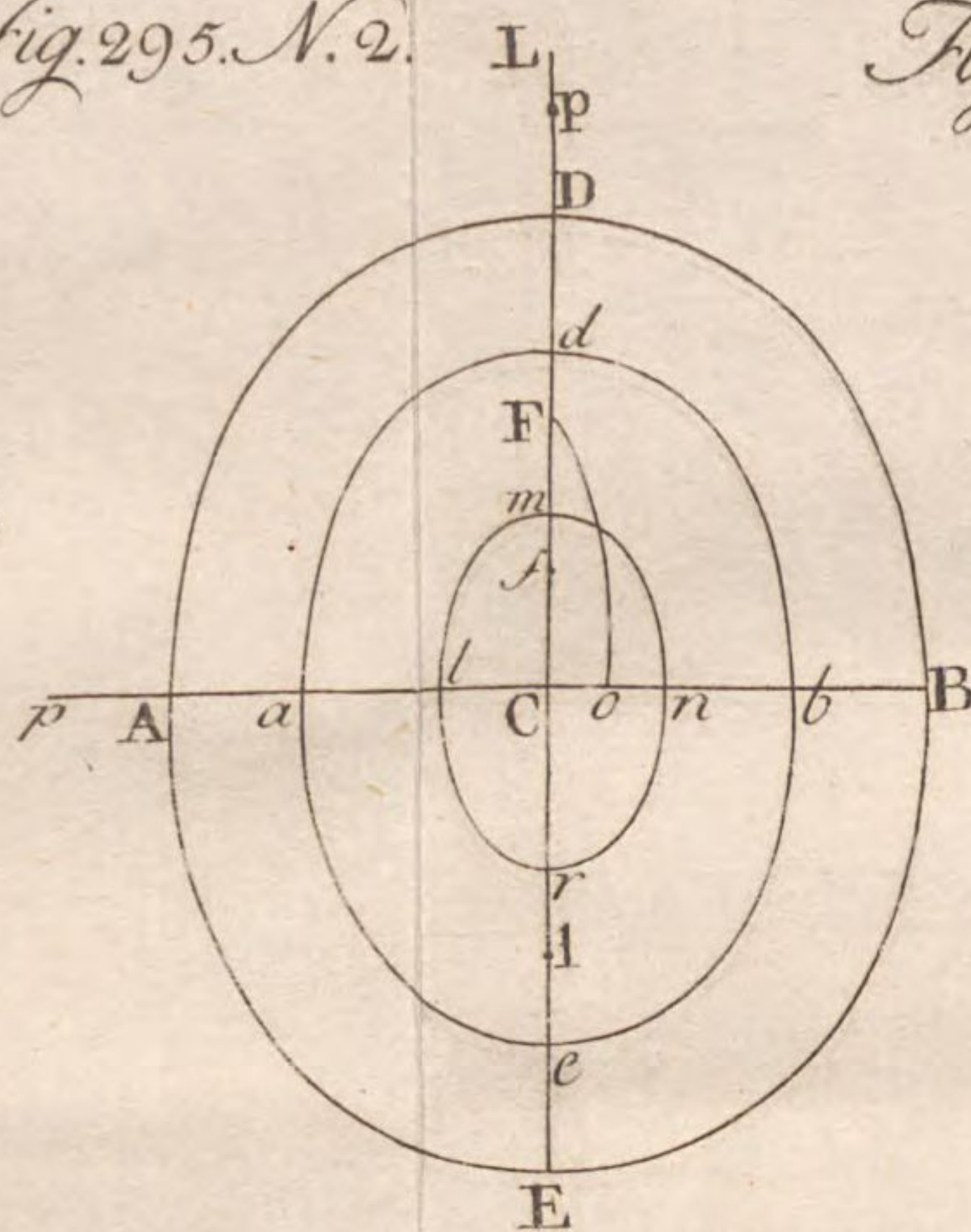


Fig. 296.

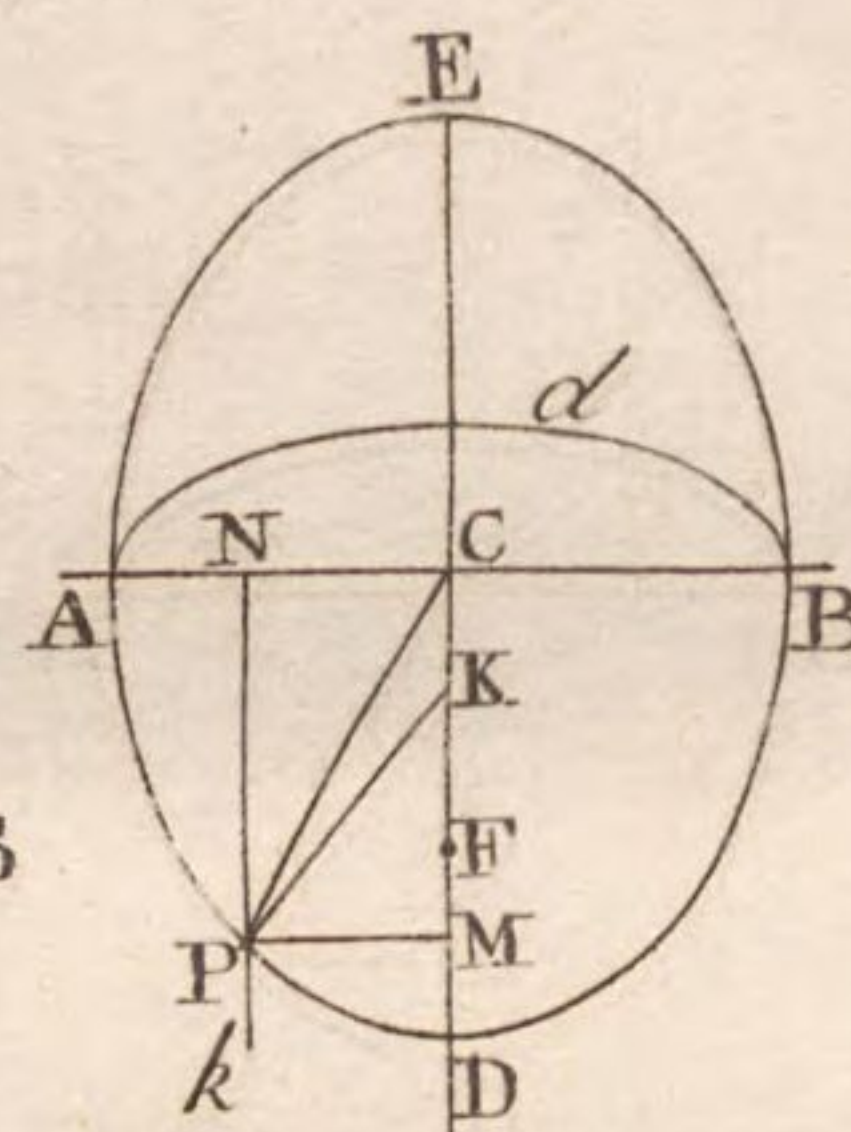


Fig. 297.

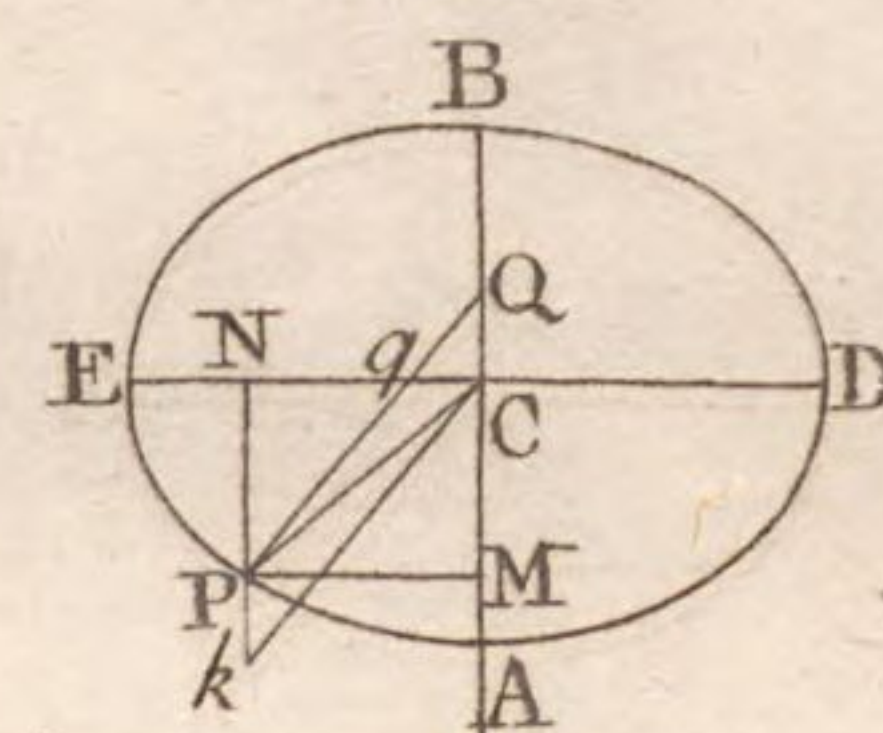


Fig. 300.

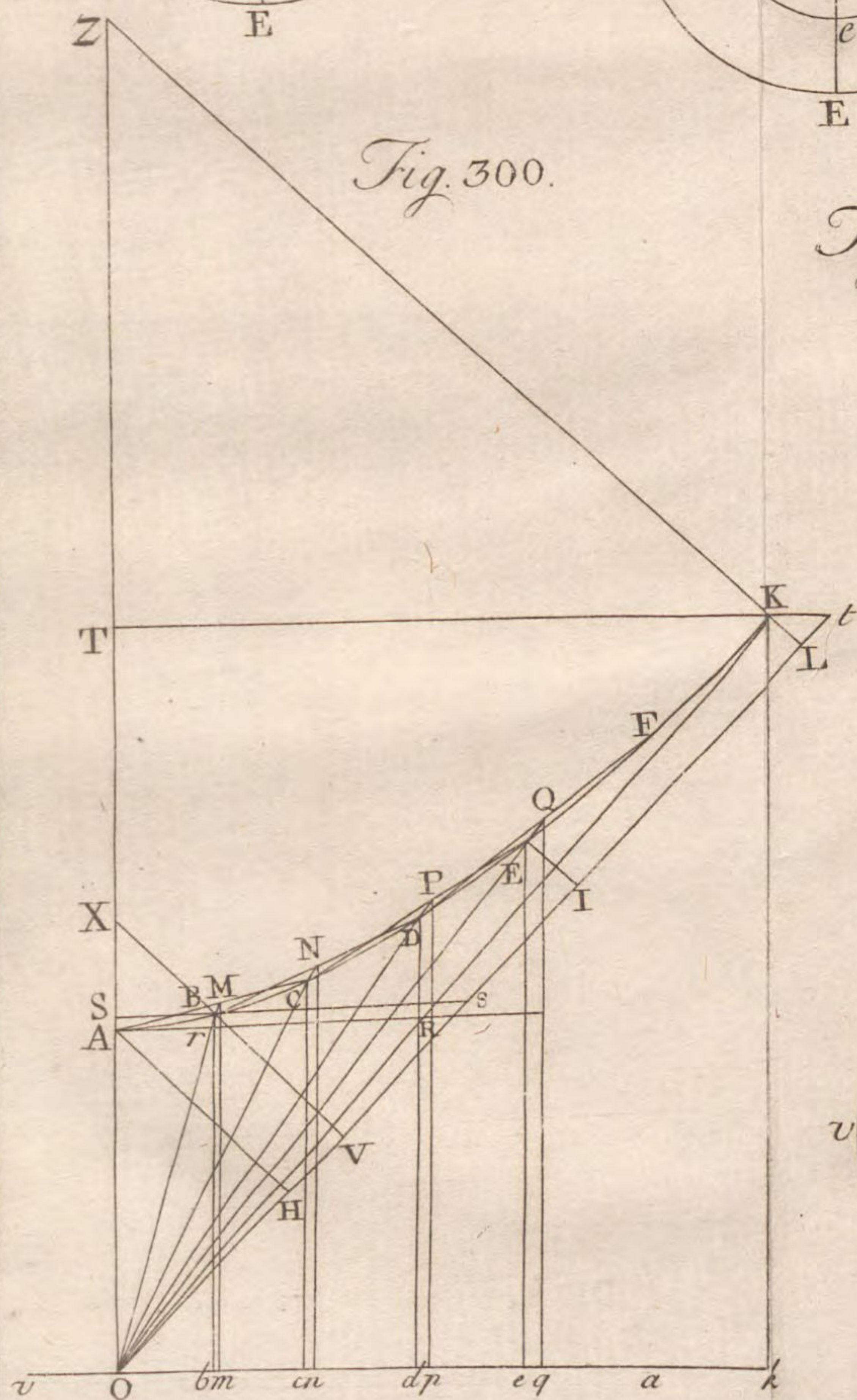


Fig. 298.

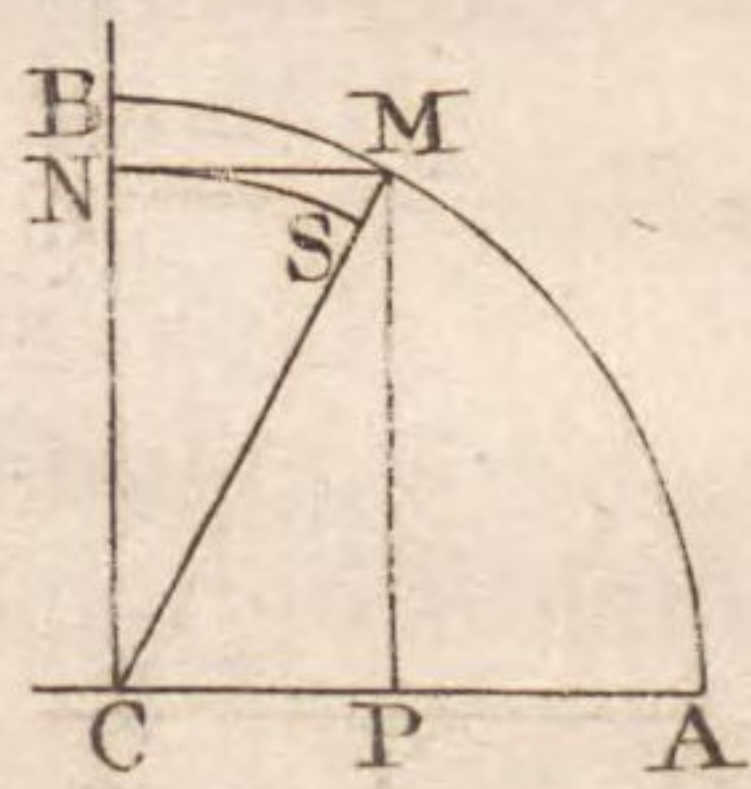


Fig. 301. N. 2. Art. 759.

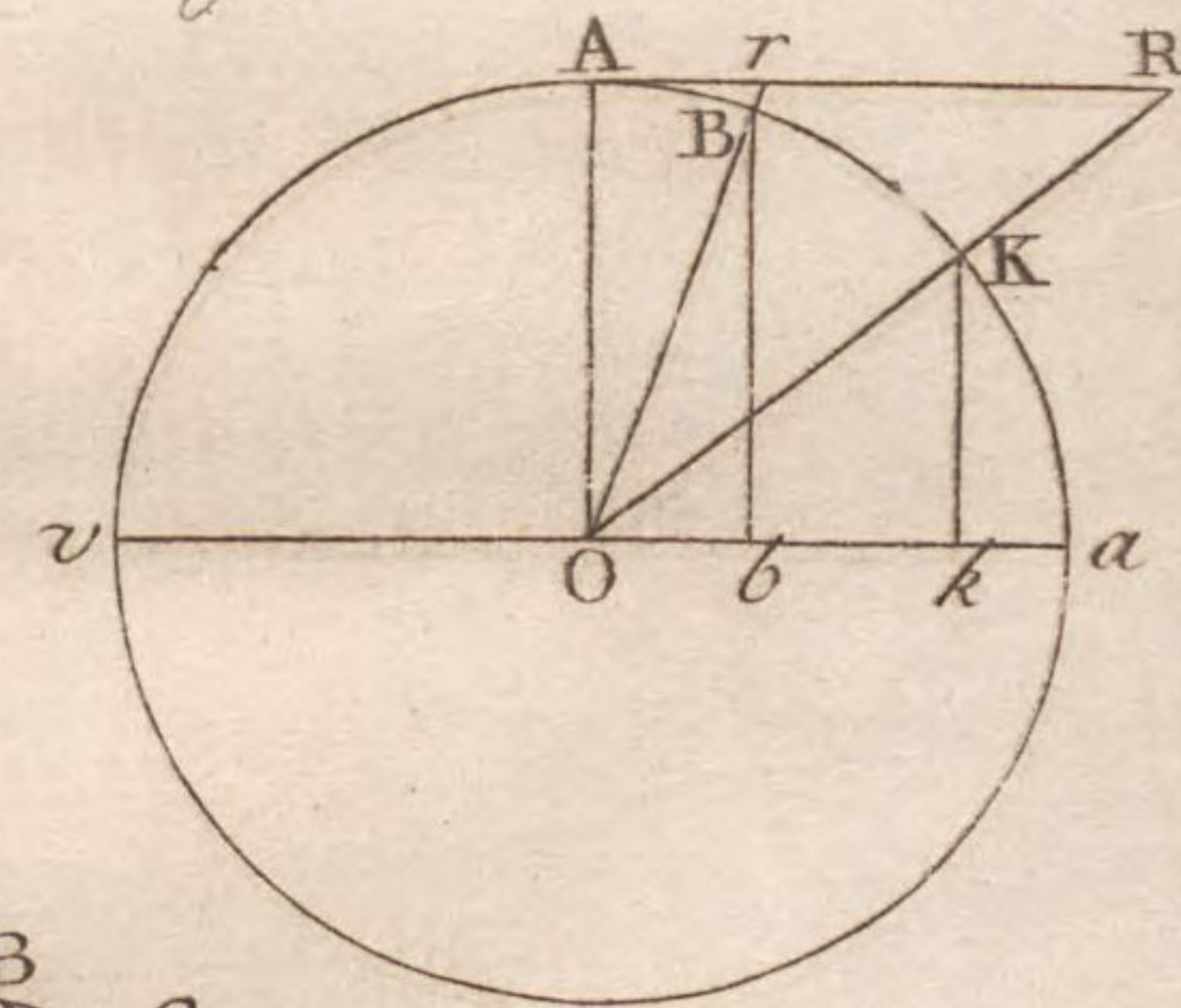


Fig. 301. N. 1.

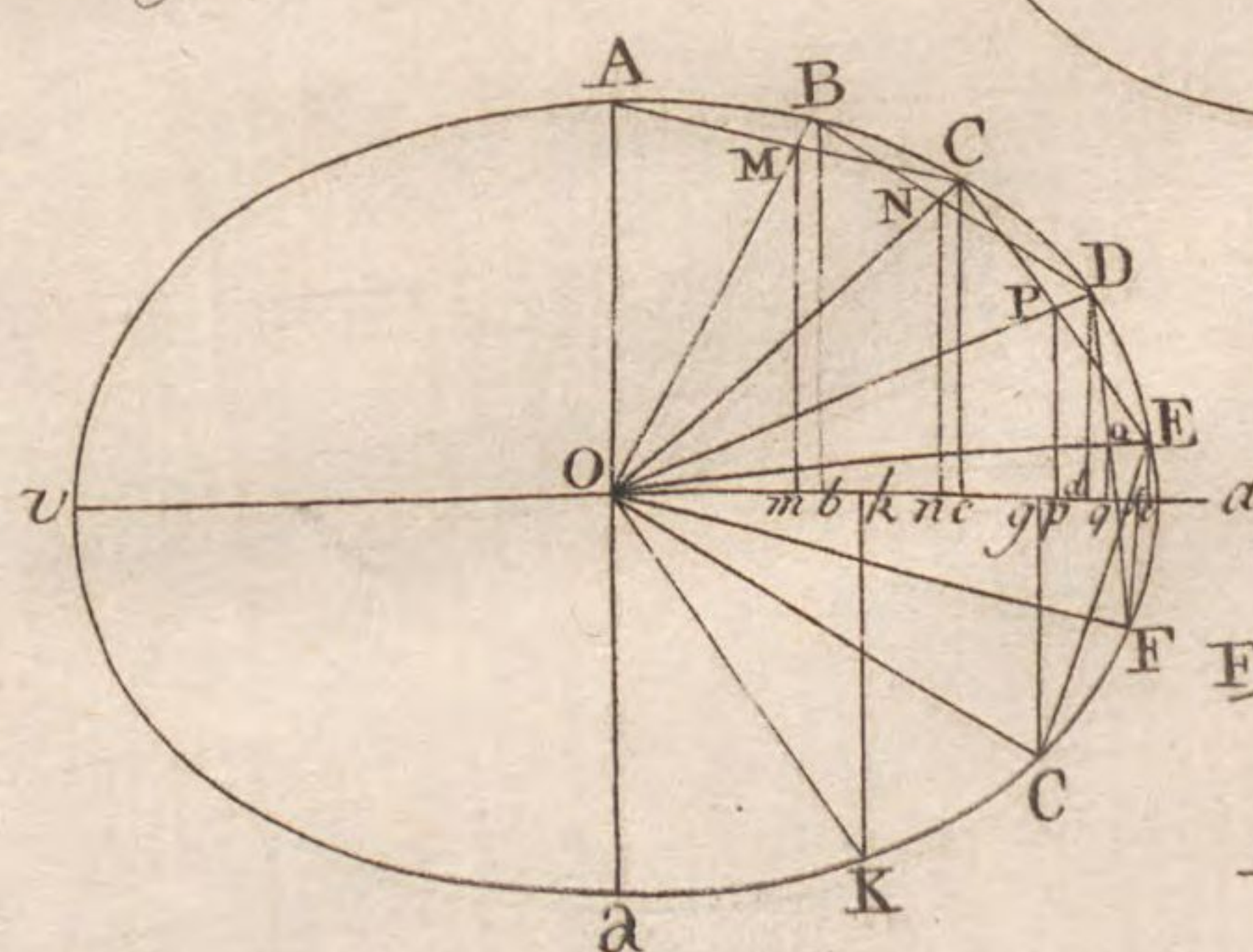
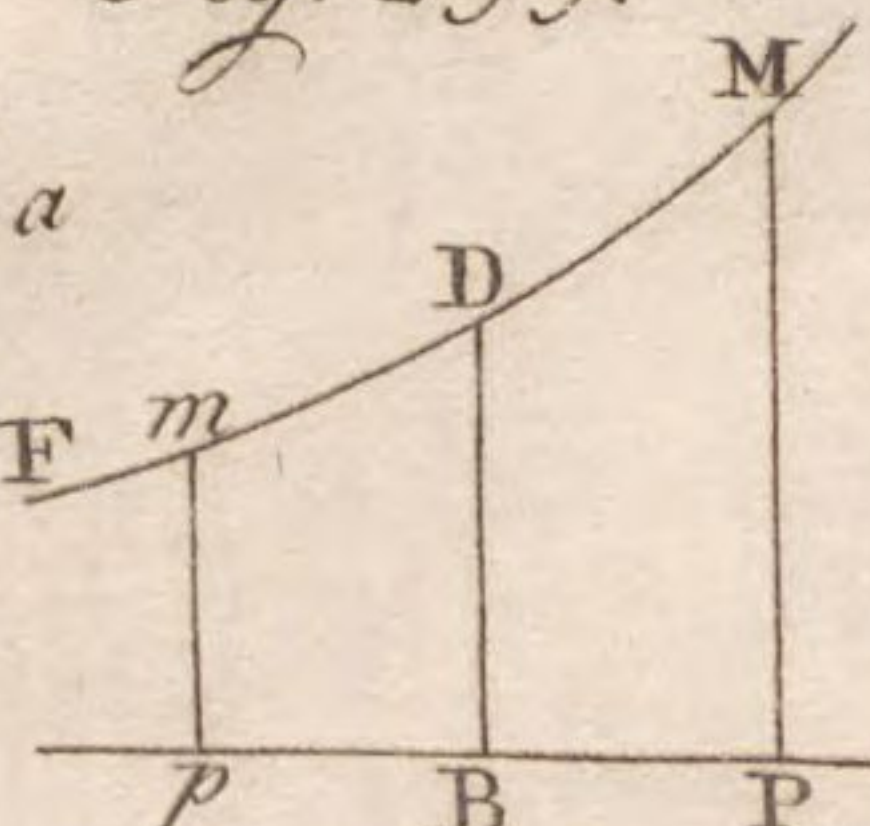


Fig. 299.



764. In like manner supposing, as above, $OA = a$, $Bb = x$, $Kk = z$, the fluxion of the ark AB (art. 747.) is $\frac{ax}{\sqrt{aa-xx}}$, and the fluxion of AK is $\frac{az}{\sqrt{aa-zz}}$. These fluxions are transformed, by supposing $p = x + \sqrt{xx-aa}$, and $q = z + \sqrt{zz-aa}$ (by art. 728.) into $\frac{ap}{p\sqrt{-1}}$ and $\frac{aq}{q\sqrt{-1}}$. Therefore since $AK = n \times AB$, $\frac{q}{p} = \frac{np}{p}$, $q = p^n \times K$, or $z + \sqrt{zz-aa} = x + \sqrt{xx-aa}^n \times K$ (because when Bb or $x = a$, then $z = a$, and consequently $a = a^n K$) $a \times \frac{x + \sqrt{xx-aa}^n}{a^n}$, as we found in art. 759. And supposing $y = x + \sqrt{xx-aa}$, the relation of z to x will be determined by exterminating y from the equations $y^{2n} - 2za^{n-1}y^n + a^{2n} = 0$, and $y^2 - 2zy + a^2 = 0$, as we demonstrated in art. 759. without making use of the imaginary sign, by shewing that the relation of z to x must be determined by the same equations in the circle and hyperbola.

765. Let the circumference of the circle be represented by **FIG. 302.** C , the radius OA by 1, the ark AK by A , and consequently AB by $\frac{A}{n}$. Let the circumference be divided into as many equal parts (beginning from the point B) BC , CD , DE , EF , FG , GB as there are units in n ; then $AB = \frac{A}{n}$, $AG = BG = \frac{C-A}{n}$, $AC = AB + BC = \frac{C+A}{n}$, $AF = \frac{2C-A}{n}$, $AD = \frac{2C+A}{n}$, and $AE = \frac{3C-A}{n}$. The co-sine Kk of the ark AK being represented by $\mp z$ as before, let the co-sines of those several arks AB , AG , AC , AF , AD , AE , &c. be represented by $\mp a$, $\mp b$, $\mp c$, $\mp d$, $\mp e$, $\mp f$, &c. respectively, where the sign of each co-sine is supposed positive or negative according

ding as it is on the same side of the diameter au with OA , or not. Then because those arcs are in the same ratio of 1 to n to the several arcs $A, C-A, C+A, 2C-A, 2C+A, 3C-A, 3C+A, \&c.$ which have all the same co-sine $Kk = \frac{1}{n}$; the several relations of z to a, z to b, z to c, z to $d, \&c.$ are found by comparing successively the equation $y^{2n} - 2zy^n + 1 = 0$, with the several quadratic equations $yy - 2ay + 1 = 0, yy - 2by + 1 = 0, yy - 2cy + 1 = 0, yy - 2dy + 1 = 0, \&c.$ and always exterminating y . From which it follows that the trinomials $yy - 2ay + 1, yy - 2by + 1, yy - 2cy + 1, yy - 2dy + 1, \&c.$ are Divisors of $y^{2n} - 2zy^n + 1$, or this last is equal to the product of those trinomials.

766. Upon OA take OP to represent y , join $PB, PC, PD,$

$PE, \&c.$ and $PB^2 = Ob^2 + Bb^2 + 2Bb \times OP + OP^2 = 1 - 2ay + yy$. In the same manner $PG^2 = 1 - 2by + yy, PC^2 = 1 - 2cy + yy, \&c.$ consequently $PB^2 \times PG^2 \times PC^2 \times PF^2 \times PD^2 \times \&c. = y^{2n} - 2zy^n + 1 = y^{2n} \pm 2Kk \times y^n + 1$. Let the arc AK or A be now supposed

FIG. 303. equal to the whole circumference C , then $BA = \frac{A}{n} = \frac{C}{n} = BG$, and G coincides with A . In this Case the point K falls on the point $A, \frac{1}{n} = Kk = OA = 1$; and $PB^2 \times PA^2 \times PC^2 \times PD^2 \times \&c. = y^{2n} - 2y^n + 1$; and $PB \times PA \times PC \times PD \times \&c. = y^n - 1$ or $1 - y^n$. From which it follows that if the circumference be divided into as many equal Parts at $A, B, C, D, \&c.$ as there are units in n , upon $OA (= 1)$ you take $OP = y$, and from P draw right lines to the other points $B, C, D, E, F, \&c.$ then the product of all those right lines, $PA \times PB \times PC \times PD \times \&c.$ will be equal to $y^n - 1$, or $1 - y^n$, that is to $OP^n - OA^n$, or $OA^n - OP^n$, according as OP is greater or less than OA ; which coincides with the first part of the elegant theorem invented by Mr. COTES, and described by Dr. SMITH, *Harmon. Mensurar. p. 114*. Let the semidiameter AO produced meet the circle in a , and if n be an even number, one of the points wherein the circumference is supposed to be divided will fall on

on a ; consequently the divisors of $1 - y^n$ will be the rectangle APa with the squares of the right lines $PB, PC, &c.$ that are on one side of Aa ; but if n be an odd number, PA (or $1 - y$) will be a simple divisor of $1 - y^n$, and the same squares of $PB, PC, &c.$ will be its other divisors. Suppose now the ark AK equal to the semi-circumference, or to $\frac{1}{2} C$; then $BA = \frac{A}{n} = \frac{C}{2n}$ = FIG. 303.

$\frac{1}{2} BG$; and in this case K falls upon a, Kk , or $Oa, = -z = 1$, or $z = -1$. From which it appears that if the circumference be divided at $B, C, D, E, &c.$ into as many equal parts as there are units in n ; and one of those parts BG being bisected in A , you join OA , and take OP upon it to represent y , then from P draw the right lines $PB, PC, PD, PE, &c.$ to the several divisions of the circumference, the product of the squares of all those lines $PB^2 \times PC^2 \times PD^2 \times PE^2 \times &c.$ will be equal to $y^{2n} + 2zy^n + 1$; and consequently $PB \times PC \times PD \times PE \times &c. = 1 + y^n = OA^n + OP^n$; which coincides with the latter part of Mr. COTES's theorem. When n is an even number the same product is that of the squares of the several right lines $PB, PC, &c.$ that are on the same side of the diameter Aa ; which are therefore the quadratic divisors of $1 + y^n$ in this case; but when n is an odd number, one of the divisions of the circumference falls upon a ; and the same squares with Pa , or $1 + y$, are the divisors of $1 + y^n$. By supposing AK equal FIG. 304. to a quadrant of the circle, or to $\frac{1}{4} C$, or $z = 0$, it will appear that the circumference being divided as before, if the ark BA be taken upon BG equal to $\frac{1}{4} BG$, and upon OA you take $OP = y$, then $PB^2 \times PC^2 \times PD^2 \times &c. = 1 + y^{2n} = OA^{2n} + OP^{2n}$. The reader will find this subject treated in a different manner, *Epist. ad amicum de Cotesii inventis* 1722.

767. In general, the circumference being divided into FIG. 300. the same number of equal parts, let P be any point in the plane of the circle, let OP meet the circumference in A , take the ark $AK = n \times AB$, and upon OA take OQ :

$K \ k \ k \ k$
 OP

OP : : OPⁿ⁻¹ : OAⁿ⁻¹, join QK; and PB² × PC² × PD² × PE² × &c. = QK² × OA²ⁿ⁻²; because Kk being supposed equal to +z, or -z, (according as Kk is on the same side of av with OA, or on a different side,) then QK² = 1 - 2z × OQ + OQ² = 1 - 2zyⁿ + y²ⁿ. In this manner Mr. COTES's theorem was rendered more general by Mr. DE MOIVRE. Hence several other propositions relating to the circle may be briefly derived. By supposing OP to coincide with OA, it appears that the product of the chords AB × AC × AD × AE × &c. = AK × OAⁿ⁻¹. For in this case OA, OP and OQ being equal, AB² × AC² × AD² × AE² × &c. = AK² × OA²ⁿ⁻² and AB × AC × AD × &c. = AK × OAⁿ⁻¹; which is demonstrated in a different manner, *Hospital. sect. coniq. lib. 10. theor. 1 & 3.*

768. The quadratic divisors of a trinomial may be likewise discovered from the common algebra. To resolve a quantity as $y^{2n} - 2zy^n + 1$ into its divisors is a problem equivalent to the resolution of the equation $y^{2n} - 2zy^n + 1 = 0$. By proceeding as is usual in the resolution of quadratic equations, $y^{2n} - 2zy^n + zz = zz - 1$, $y^n - z = \pm \sqrt{zz-1}$; and the divisors are $y^n - z + \sqrt{zz-1}$ and $y^n - z - \sqrt{zz-1}$, the product of which is $y^{2n} - 2zy^n + 1$. When z is less than 1, $\sqrt{zz-1}$ is imaginary, and those divisors involve imaginary expressions. But we are not thence to conclude that other divisors cannot be assigned in this case, which may involve real quantities only. It is obvious that $\overline{y-a} \times \overline{y-b} \times \overline{y-c} \times \overline{y-d}$ may be resolved into several different pairs of quadratic divisors, as into $\overline{y-a} \times \overline{y-b}$ and $\overline{y-c} \times \overline{y-d}$, or into $\overline{y-a} \times \overline{y-c}$ and $\overline{y-b} \times \overline{y-d}$; and though the first two may involve the imaginary symbol, the latter may involve no quantities but such as are real. Thus supposing $y^4 - 2zy^2 + 1 = 0$, we have $y^2 = z + \sqrt{zz-1}$ or $z - \sqrt{zz-1}$; and the four simple divisors, (by extracting the square root again) are in this case $y + \sqrt{z + \sqrt{zz-1}}$, $y - \sqrt{z + \sqrt{zz-1}}$, $y +$

$y + \sqrt{z - \sqrt{zz-1}}$, and $y - \sqrt{z - \sqrt{zz-1}}$. The product of the first and second gives $yy - z - \sqrt{zz-1}$; and the product of the third and fourth gives $yy - z + \sqrt{zz-1}$, the same intermediate divisors from which the simple divisors were derived; both of which involve an imaginary quantity when z is less than 1. But the product of the first and third gives $yy + \sqrt{z + \sqrt{zz-1}} + \sqrt{z - \sqrt{zz-1}}$ $\times y + 1 =$ (because the square of $\sqrt{z + \sqrt{zz-1}} + \sqrt{z - \sqrt{zz-1}}$ is $z + \sqrt{zz-1} + 2 + z - \sqrt{zz-1} = 2z + 2$) $yy + \sqrt{2z+2} \times y + 1$ or $yy - \sqrt{2z+2} \times y + 1$; and the product of the third and fourth agrees with these. Thus we find $y^4 - 2zy^2 + 1 = yy + \sqrt{2z+2} \times y + 1 \times yy - \sqrt{2z+2} \times y + 1$. And these divisors may involve no imaginary quantity, tho' z be supposed negative and less than unit. By continuing the resolution, till we have the simple divisors; and then compounding those divisors together variously, quadratic divisors may be formed in this manner, some of which will have all their coefficients real when z is greater than unit, and others when z is less than unit; and we are not to conclude that no real quadratic divisors can be assigned, because those of one combination are imaginary. The latter quadratic divisors are likewise found by resolving the equation $y^4 - 2zy^2 + 1 = 0$ in a manner somewhat different from the usual way of proceeding; for since $y^4 + 1 = 2zy^2$, complete the square on the first side of the equation by adding the middle term $+ 2y^2$, and $y^4 + 2y^2 + 1 = + 2zy^2 + 2y^2$; consequently by extracting the square-root, $y^2 + 1 = + \sqrt{2z+2} \times y$; and the quadratic divisors are $y^2 - \sqrt{2z+2} \times y + 1$ and $y^2 + \sqrt{2z+2} \times y + 1$, as before. By the same method the divisors of $y^{2n} - 2zy^n + 1$ are $y^n \mp \sqrt{2z+2} \times y^{\frac{n}{2}} + 1$, which may be again resolved in the same manner into divisors of inferior dimensions; and by a continual bisection of the exponent, when n is any power of the number 2, we may at length find the quadratic divisors. But this last method is not applicable when n is any other number.

769. Supposing therefore $y^{2n} - 2zy^n + 1 = 0$, as before; and consequently $y^n = z + \sqrt{zz-1}$ or $z - \sqrt{zz-1}$; then by
K k k k 2
re-

evolution $y = \sqrt[n]{z + \sqrt{zz-1}}$ or $\sqrt[n]{z - \sqrt{zz-1}}$; consequently two of the simple divisors are $y = \sqrt[n]{z + \sqrt{zz-1}}$ and $y = \sqrt[n]{z - \sqrt{zz-1}}$; and, the quadratic divisor arising from their multiplication by each other being $yy = \sqrt[n]{z + \sqrt{zz-1}} + \sqrt[n]{z - \sqrt{zz-1}}$ $\times y + 1$, suppose the coefficient of the middle term to be $2x$, or this quadratic divisor to be $yy = 2xy + 1$, and $2x = \sqrt[n]{z + \sqrt{zz-1}} + \sqrt[n]{z - \sqrt{zz-1}}$. By comparing this equation with that which was deduced in art. 759, it will appear that, when z is less than 1, if z be the co-sine of a circular ark A , then x will be the co-sine of an ark equal to $\frac{A}{n}$. Or if we suppose $z + \sqrt{zz-1} = p$ and $z - \sqrt{zz-1} = q$, and consequently $2z = p + q$, $x = \frac{p^{\frac{1}{n}} + q^{\frac{1}{n}}}{2}$, $\sqrt{xx-1} = (\text{because } pq = 1)$ $\frac{p^{\frac{1}{n}} - q^{\frac{1}{n}}}{2}$, $x + \sqrt{xx-1} = p^{\frac{1}{n}}$ and $x - \sqrt{xx-1} = q^{\frac{1}{n}}$; so that $2z = x + \sqrt{xx-1} + x - \sqrt{xx-1}$, the same equation that we found in art 759, for the co-sines; which, expanded as above, will be found always to agree with those by which the relations of the co-sines are determined by the common methods. But let us now proceed to shew how fluents, or areas, are measured by circular arks and logarithms; and first when the ordinates are expressed by rational quantities.

770. Let it be required to assign the fluent of $\frac{y^n}{y-a}$, n being any integer positive number. It was shown in art. 709, that if $y^{n-1} + y^{n-2}a + y^{n-3}a^2 \dots a^{n-1}$ be multiplied by $y - a$, the product will be $y^n - a^n$. Therefore $\frac{y^n}{y-a} = y^{n-1} + \frac{a^n}{y-a}$

$ay^{n-2} + a^2y^{n-3} \dots + a^{n-1}y + \frac{a^n y}{y-a}$; consequent-

ly the fluent of $\frac{y^n}{y-a}$ (by art. 737. and 740.) is $\frac{y^n}{n} + \frac{ay^{n-1}}{n-1}$

$+ \frac{a^2y^{n-2}}{n-2} \dots + a^{n-1}y + \frac{a^n}{M} \times \log. y-a$. In the same man-

ner $y^{n-1} - y^{n-2}a + y^{n-3}a^2 \dots - a^{n-1} = \frac{y^n - a^n}{y+a}$.

Therefore the fluent of $\frac{y^n}{y+a}$ is $\frac{y^n}{n} - \frac{ay^{n-1}}{n-1} + \frac{a^2y^{n-2}}{n-2} \dots$

$- \frac{a^n}{M} \times \log. y+a$.

771. Any integer number being represented by n , the fluent of $\frac{y^n}{aa+yy}$ is expressed by a circular ark, or logarithm, (with algebraic quantities,) according as n is an even or odd number. For it appears, as in the last article, that when n is an even positive number, if $y^{n-2} - a^2y^{n-4} + a^4y^{n-6} - a^6y^{n-8} + \&c.$ be multiplied by $y^2 + a^2$, the product will be $y^n - a^n$, or $y^n + a^n$, according as $\frac{1}{2}n$ is an even or odd number.

Therefore $\frac{y^n}{yy+aa} = y^{n-2} - a^2y^{n-4} + a^4y^{n-6} \dots$

$+ \frac{a^n y}{yy+aa}$; consequently if A represent the ark whose tan-

gent is equal to y , the radius being equal to a (so that $A = \frac{aay}{yy+aa}$, by art. 744.) the fluent of $\frac{y^n}{yy+aa}$ will be equal to

$\frac{y^{n-1}}{n-1} - \frac{a^2y^{n-3}}{n-3} + \frac{a^4y^{n-5}}{n-5} \dots + a^{n-2} \times A$. When n is an

odd affirmative number, suppose it equal to $m+1$; and, by what has

has been shown, $\frac{y^{m+1}}{yy+aa} = y^{m-1} - y^{m-3}a^2 + y^{m-5}a^4 \dots$

$+ \frac{a^m yy}{yy+aa}$; the fluent of which (because the fluent of $\frac{yy}{aa+yy}$ is $\frac{\log. \sqrt{aa+yy}}{M}$) is $\frac{y^m}{m} - \frac{a^2 y^{m-2}}{m-2} \dots + \frac{a^m}{M} \times \log. \sqrt{aa+yy}$
 $= \frac{y^{n-1}}{n-1} - \frac{a^2 y^{n-3}}{n-3} + \frac{a^4 y^{n-5}}{n-5} \dots + \frac{a^{n-1}}{M} \times \log. \sqrt{aa+yy}$

By supposing $y = \frac{aa}{z}$, $\frac{y^{n-1}}{aa+yy}$ is transformed into $-\frac{z^{n-1}}{aa+zz} \times$

$\frac{1}{a^{2n}}$, the fluent of which (by what has been shown) is expressed by a circular ark or logarithm according as n is an even or odd number. By supposing $z = a \times \frac{y-a}{y+a}$, the fluxion $\frac{aay}{yy-aa}$

is transformed into $\frac{az}{2z}$ by art. 728. consequently the fluent is

$\frac{a}{2M} \times \log. a \times \frac{y-a}{y+a}$; and the fluent of $\frac{yy}{yy-aa}$ being equal to $\frac{\log. \sqrt{yy-aa}}{M}$, it easily follows that when n is any integer num-

ber, the fluent of $\frac{y^n}{yy-aa}$ is expressed by logarithms and algebraic quantities.

772. Let $\frac{y}{aa+2by+yy} = \dot{Q}$, and let it be required to find the fluent of $\dot{Q}$. By supposing $y+b=z$, and consequently

$y=z, yy+2by+aa=zz+aa-bb$, $\dot{Q} = \frac{z}{zz+aa-bb}$, and the fluent Q is expressed by a circular ark, or logarithm, according as a is greater or less than b , by the last article; and if $a=b$ the fluent is $-\frac{1}{z} + K$ by art. 737. The fluxion

$\frac{yy}{aa+2by+yy}$ is transformed into $\frac{zz-bz}{zz+aa-bb}$, by the same substitution;

tion; and the fluent may be found by the last article. Or, supposing $\sqrt{aa + 2by + yy} = u$, because $\frac{yy + by}{aa + 2by + yy}$ is equal to $\frac{u}{u}$ by art.

728. it follows that the fluent of $\frac{yy}{aa + 2by + yy}$ is $\frac{\log. u}{M} - b Q =$

$\frac{\log. \sqrt{aa + 2by + yy}}{M} - bQ$. The fluent of $\frac{y^n}{aa + 2by + yy}$ is found

(when n is any integer positive number) by dividing y^n by $yy + 2by + aa$, and continuing the operation till the remainder be of the form $Ay + B$, (where A and B represent invariable coefficients) multiplying each term of the quotient by y , finding the fluent of each product by art. 737. and determining the fluent of

$\frac{Ayy + By}{aa + 2by + yy}$ by this article. The fluxion $\frac{y^{-n}}{1 + 2by + yy}$ is trans-

formed, by supposing $y = \frac{1}{z}$, into $\frac{-z^n}{1 + 2bz + zz}$, and the fluent

is found as before. It appears therefore that, the ordinate being expressed by a fraction, if the denominator be any quadratic trinomial $1 + 2by + yy$, and the numerator consist of terms that involve any powers of y and invariable quantities; and the exponents of those powers of y be integers; the fluent may be assigned by circular arks or logarithms with algebraic quantities. And any fluxion $P\dot{y}$ being proposed, if P can be resolved into any number of fractions of this form, the fluent of $P\dot{y}$ can be assigned in like manner.

773. What was demonstrated in art. 715, and 717. or in 728, and 729. is often of use for resolving an ordinate into such fractions. For example, as when $p = xyz \times \&c.$ it follows that

$\frac{p}{p} = \frac{x}{x} + \frac{y}{y} + \frac{z}{z} + \&c.$ so if we resolve $1 + y^n$ (supposing n to be an even number) into its quadratic divisors $1 - 2ay + yy$, $1 - 2by + yy$, $1 - 2cy + yy$, $\&c.$ according to

art. 765, it follows that $\frac{y^{n-1}}{1 + y^n} = \frac{2yy - 2ay}{1 - 2ay + yy} + \frac{2yy - 2by}{1 - 2by + yy} + \&c.$

$$+ \frac{2yy - 2cy}{1 - 2cy + yy} + \&c. \text{ and consequently } \frac{ny^n}{1 + y^n} = \frac{2yy - 2ay}{1 - 2ay + yy} + \frac{2yy - 2by}{1 - 2by + yy} + \&c. = (\text{because } \frac{2yy - 2ay}{1 - 2ay + yy} = \frac{2yy - 4ay + 2 + 2ay - 2}{1 - 2ay + yy})$$

$$= 2 + \frac{2ay - 2}{1 - 2ay + yy} + \frac{2by - 2}{1 - 2by + yy} + \frac{2cy - 2}{1 - 2cy + yy} + \&c.$$

so that the fluent of $\frac{y^n}{1 + y^n}$ is equal to y added to the fluents of the several fluxions $\frac{2ay - 2}{1 - 2ay + yy} \times \frac{y}{n}$, $\frac{2by - 2}{1 - 2by + yy} \times \frac{y}{n}$, $\&c.$ which are found by the last article.

774. The same method serves for investigating briefly the first four propositions of Mr. DE MOIVRES *miscel. analyt. lib. I.* Suppose first n to be an even number, and since $1 + y^n = 1 - 2ay + yy \times 1 - 2by + yy \times 1 - 2cy + yy \times \&c.$ it follows, dividing $1 + y^n$ by y^n , and each quadratic divisor by yy , that

$$\frac{1 + y^n}{y^n} = 1 - 2ay^{-1} + y^{-2} \times 1 - 2by^{-1} + y^{-2} \times \&c.$$

Therefore, as when $\frac{p}{q} = xyzn \times \&c.$ $\frac{p}{p} - \frac{q}{q} = \frac{x}{x} + \frac{y}{y} + \frac{z}{z} + \&c.$ (by art. 728.) so in this Case $\frac{ny^n}{1 + y^n} - \frac{ny}{y} (=$

$$-\frac{ny}{1 + y^n} \times \frac{1}{y}) = \frac{2ay^{-2}y - 2y^{-3}y}{1 - 2ay^{-1} + y^{-2}} + \frac{2by^{-2}y - 2y^{-3}y}{1 - 2by^{-1} + y^{-2}} + \&c.$$

&c. or (dividing both sides by $-yy^{-1}$) $\frac{n}{1 + y^n} = \frac{2 - 2ay}{1 - 2ay + yy} + \frac{2 - 2by}{1 - 2by + yy} + \frac{2 - 2cy}{1 - 2cy + yy} ;$ which is the first of those propositions.

When n is an odd number, then, besides the quadratic

Fig. 304. divisors of $1 + y^n$ (which according to art. 766. are PB^2 , PC^2 , PD^2

PD², &c.) there is a simple divisor Pa = 1 + y; and it appears that in this case $\frac{n}{1+y^n} = \frac{1}{1+y} + \frac{2-2ay}{1-2ay+yy} + \frac{2-2by}{1-2by+yy} + \&c.$ When *n* is an even number, the rectangle APa, or 1-yy, is one of the divisors of 1-yⁿ (by art. 766.) and the other quadratic divisors PB², PC², PD², &c. being expressed by 1-2ay+yy, 1-2by+yy, &c. as before, it appears in the same manner, that in this case $\frac{n}{1-y^n} = \frac{2}{1-yy} + \frac{2-2ay}{1-2ay+yy} + \frac{2-2by}{1-2by+yy} + \&c.$ And when *n* is an odd number, PA or 1-y being one of the divisors of 1-yⁿ (by art. 766.) we shall find in the same manner that $\frac{n}{1-y^n} = \frac{1}{1-y} + \frac{2-2ay}{1-2ay+yy} + \frac{2-2by}{1-2by+yy} + \&c.$ The ordinate $\frac{1}{1+y^n}$ being resolved in this manner into fractions with quadratic denominators, the area or the fluent of $\frac{y}{1+y^n}$ (and consequently of $\frac{y^m}{1+y^n}$ when *m* is any integer number) is reduced to circular arks or logarithms with algebraic quantities by art. 772. The ordinate $\frac{1}{e+fy^n+gy^{2n}}$ is resolved into fractions of the same kind by this method, when *ff* is greater than 4*eg*, that is, when the roots of the quadratic equation $y^{2n} + \frac{fy^n}{g} + \frac{e}{g} = 0$ are real. For supposing those roots to be -Rⁿ and -rⁿ, or $e+fy^n+gy^{2n} = g \times y^n + R^n \times y^n + r^n$, let $y^n + R^n$ and $y^n + r^n$ be resolved into their respective divisors (art. 766.) RR-2Ay+yy, &c. and rr-2ay+yy, &c. and by what has been shewn $\frac{n}{r^n+y^n} - \frac{n}{R^n+y^n}$

L I I I

or

$$\text{or } \frac{R^n - r^n}{e + fy^n + gy^{2n}} \times ng = \frac{1}{r^n} \times \frac{2rr - 2ay}{rr - 2ay + yy} + \&c. - \frac{1}{R^n} \times \frac{2RR - 2Ay}{RR - 2Ay + yy} - \&c.$$

775. When z is less than 1, let $y^{2n} - 2zy^n + 1$ be resolved (by art. 766.) into its divisors $1 - 2ay + yy$, $1 - 2by + yy$, &c. Then (z being supposed invariable) it follows as in art. 728. that

$$\frac{ny^{2n-1} - nzy^{n-1}}{y^{2n} - 2zy^n + 1} = \frac{y-a}{1-2ay+yy} + \frac{y-b}{1-2by+yy} + \&c. \text{ or (multiplying}$$

$$\text{by } \frac{z}{y^{n-1}}) \frac{nzy^n - nzz}{y^{2n} - 2zy^n + 1} = \frac{y-a}{1-2ay+yy} \times \frac{z}{y^{n-1}} + \frac{y-b}{1-2by+yy} \times$$

$$\frac{z}{y^{n-1}} + \&c. \text{ Because } \frac{y^{2n} - 2zy^n + 1}{y^{2n}} = 1 - 2ay^{-1} + y^{-2} \times$$

$$1 - 2by^{-1} + y^{-2} \times \&c. \text{ it follows (as in art. 728.) that}$$

$$\frac{n - nzy^n}{y^{2n} - 2zy^n + 1} = \frac{1-ay}{1-2ay+yy} + \frac{1-by}{1-2by+yy} + \&c. \text{ The sum of}$$

$$\text{those equations gives } \frac{n - nzz}{y^{2n} - 2zy^n + 1} = \frac{y-a \times zy^{-n+1}}{1-2ay+yy} + \frac{1-ay}{1-2ay+yy}$$

$$+ \frac{y-b \times zy^{-n+1}}{1-2by+yy} + \frac{1-by}{1-2by+yy} + \&c. \text{ and it follows from art.}$$

$$772. \text{ that the fluent of } \frac{y}{y^{2n} - 2zy^n + 1} \text{ is assignable by cir-}$$

cular arks and logarithms with algebraic quantities.

776. By a similar application of what was shewn in art. 728. if we suppose $xx - Ax + B = \frac{x-a}{x-b}$, the fraction

$\frac{x}{xx - Ax + B}$ is resolved into fractions that shall have the simple

divisors $x-a$, $x-b$ for their respective denominators with invariable coefficients. For since $xx - Ax + B = \frac{x-a}{x-b}$, it follows

follows (art. 728.) that $\frac{2xx - Ax}{xx - Ax + B} = \frac{x}{x-a} + \frac{x}{x-b}$; and because $1 - Ax^{-1} + Bx^{-2} = 1 - ax^{-1} \times 1 - bx^{-1}$, it follows that $\frac{Axx - 2Bx}{xx - Ax + B} = \frac{ax}{x-a} + \frac{bx}{x-b}$. From these two equations we have $\frac{x}{xx - Ax + B} = \frac{\frac{1}{2}A - a}{2B - \frac{1}{2}AA} \times \frac{x}{x-a} + \frac{\frac{1}{2}A - b}{2B - \frac{1}{2}AA} \times \frac{x}{x-b}$ (because $A = a + b$ and $B = ab$, by the known properties of equations) $\frac{1}{a-b} \times \frac{x}{x-a} + \frac{1}{b-a} \times \frac{x}{x-b}$. Hence $\frac{1}{xx - Ax + B} = \frac{1}{a-b} \times \frac{1}{x-a} + \frac{1}{b-a} \times \frac{1}{x-b}$. Therefore if $x^3 - Ax^2 + Bx - C = \overline{x-a} \times \overline{x-b} \times \overline{x-c}$, it follows that $\frac{1}{x^3 - Ax^2 + Bx - C} = \frac{1}{\overline{x-a} \times \overline{x-b} \times \overline{x-c}} \times \frac{1}{x-c}$ (by what has been shewn) $\frac{1}{a-b \times \overline{x-a} \times \overline{x-c}} + \frac{1}{b-a \times \overline{x-b} \times \overline{x-c}}$ (by resolving each of these last fractions in the same manner) $\frac{1}{a-b \times \overline{a-c} \times \overline{x-a}} + \frac{1}{a-b \times \overline{c-a} \times \overline{x-c}} + \frac{1}{b-a \times \overline{b-c} \times \overline{x-b}} + \frac{1}{b-a \times \overline{c-b} \times \overline{x-c}} = \frac{1}{a-b \times \overline{a-c} \times \overline{x-a}} + \frac{1}{b-a \times \overline{b-c} \times \overline{x-b}} + \frac{1}{c-a \times \overline{c-b} \times \overline{x-c}}$. The continuation of those theorems is manifest, the coefficient of $x-b$, for example, being always the product of the differences $b-a, b-c, &c.$ by which the root b exceeds the other roots of the equation $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \&c. = 0$. This subject is considered by Mr. LEIBNITZ, *Act Lips.* 1702, and Mr. DE MOIVRE, *Phil. Transf.* N. 373, &c. 777. But these fractions are briefly discovered in the following manner. Suppose $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = \overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d} \times \&c.$ and let this product be represented by P; let Q represent the product of all the simple divisors, the first $x-a$ excepted; that is, let $Q = \overline{x-b} \times \overline{x-c} \times \overline{x-d} \times \&c.$

$\times \overline{x-d} \times \&c.$ Suppose $a, b, c, d, \&c.$ to be unequal and r being any integer positive number less than n , suppose $\frac{x^r}{p}$ or

$$\frac{x^r}{x^n - Ax^{n-1} + Bx^{n-2} - \&c.} = \frac{L}{x-a} + \frac{M}{x-b} + \frac{N}{x-c} + \&c.$$

where $L, M, N, \&c.$ represent the invariable coefficients that are to be determined. By reducing those fractions to a common denominator, and multiplying by P or $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \&c.$ we have $x^r = LQ + MQ \times \frac{x-a}{x-b} + NQ \times \frac{x-a}{x-c} + \&c.$ Then by supposing $x=a$, or $x-a=0$, we find that x^r (i. e. in this case a^r) is equal to LQ , or that $a^r = L \times \overline{a-b} \times \overline{a-c} \times \overline{a-d} \times \&c.$ and $L = \frac{a^r}{\overline{a-b} \times \overline{a-c} \times \overline{a-d} \times \&c.}$ In the same

manner by supposing $x=b$, we find $M = \frac{b^r}{\overline{b-a} \times \overline{b-c} \times \overline{b-d} \times \&c.}$

The other coefficients of the fractions into which $\frac{x^r}{p}$ is to be resolved, are expressed by similar values.

778. Because $P = Q \times \overline{x-a}$, it follows that $\dot{P} = \dot{Q} \times \overline{x-a} + x\dot{Q}$; and when $x=a$, $\dot{P} = x\dot{Q}$, or $\dot{Q}$ (which in this case is equal to $\overline{a-b} \times \overline{a-c} \times \overline{a-d} \times \&c.) = \frac{\dot{P}}{x} = na^{n-1} - \overline{n-1} \times$

$Aa^{n-2} + \overline{n-2} \times Ba^{n-3} - \&c.$ Therefore $L = \frac{a^r}{Q} =$

$\frac{a^r}{na^{n-1} - \overline{n-1} \times Aa^{n-2} + \overline{n-2} \times Ba^{n-3} - \&c.}$ In the same

manner $M = \frac{b^r}{nb^{n-1} - \overline{n-1} \times Ab^{n-2} + \overline{n-2} \times Bb^{n-3} - \&c.}$

and the values of the other coefficients are similar. The rule
for

for finding the coefficient in any of those fractions (as in

$\frac{N}{x-c}$) into which $\frac{x^r}{x^n - Ax^{n-1} + Bx^{n-2} - \&c.}$ is to be resol-

ved, is, substitute c for x in the numerator x^r , find the fluxion of the denominator $x^n - Ax^{n-1} + \&c.$ which being divided by $\dot{x}$, and c being substituted for x in the quotient, you have the denominator of the value of N .

779. Let it be required to resolve $\frac{x^r}{x^{2n} - Ax^{2n-1} + Bx^{2n-2} - \&c.}$

into fractions that shall have quadratic denominators, where r is supposed to be any integer and positive number less than $2n$; and the denominator (P) to be the product of the quadratic divisors $xx - 2ax + gg$, $xx - 2bx + hh$, $xx - 2cx + kk$, $\&c.$

suppose $\frac{x^r}{P} = \frac{L-lx}{xx-2ax+gg} + \frac{M-mx}{xx-2bx+hh} + \frac{N-nx}{xx-2cx+kk} +$

$\&c.$ Let Q represent the product of all the quadratic divisors, the first $xx - 2ax + gg$ excepted; that is, let $P = xx - 2ax + gg \times Q$. Then by reducing the fractions to a common denominator,

$\frac{L-lx}{xx-2ax+gg} \times Q + \frac{M-mx}{xx-2bx+hh} \times Q + \frac{N-nx}{xx-2cx+kk} \times Q +$

$\&c. = x^r$. Let e and f be the roots of the equation $xx - 2ax + gg = 0$. Let M be the value of Q when $x = e$, and N its Value when $x = f$. Then substituting e for x , $xx - 2ax + gg$, and all the terms that are multiplied by it, vanish; consequently

$\frac{L-le}{1} \times M = e^r$. In the same manner, by substituting f for

x , $\frac{L-lf}{1} \times N = f^r$. Hence $L = \frac{e^r f}{M \times f - e} - \frac{f^r e}{N \times f - e} =$ (be-

cause $ef = gg$) $\frac{gg}{f-e} \times \frac{e^{r-1}}{M} - \frac{f^{r-1}}{N}$. Because $P = xx - 2ax + gg$

$\times Q$, it follows (by taking the fluxions) that $\dot{P} = 2x - 2a$
 $\times \dot{Q} + \dot{Q} \times xx - 2ax + gg$; and by substituting e for x , $\frac{\dot{P}}{\dot{Q}}$ (which in

this

this case is $2ne^{2n-1} - \overline{2n-1} \times Ae^{2n-2} + \overline{2n-2} \times Be^{2n-3} - \&c.)$
 $= \overline{2e-2a} \times M = (\text{because } e+f=2a, \text{ and } e-f=2e-2a,)$
 $\overline{e-f} \times M.$ In the same manner $N \times \overline{f-e} = 2nf^{2n-1} - \overline{2n-1}$

$$\times Af^{2n-2} + \&c. \text{ Therefore } L = \frac{ggf^{r-1}}{2nf^{2n-1} - \overline{2n-1} \times Af^{2n-2} + \&c.}$$

$$\frac{gge^{r-1}}{2ne^{2n-1} - \overline{2n-1} \times Ae^{2n-2} + \&c.} \text{ And in like manner we}$$

$$\text{shall find } l = \frac{-e^r}{M \times \overline{e-f}} - \frac{f^r}{N \times \overline{f-e}} = \frac{-e^r}{2ne^{2n-1} - \overline{2n-1} \times Ae^{2n-2} + \&c.}$$

$\frac{f^r}{2nf^{2n-1} - \overline{2n-1} \times Af^{2n-2} + \&c.}$. The values of the coefficients M and m, N and n, &c. are similar.

780. Let gg, bb, kk, &c. the last terms of the quadratic divisors be all equal to each other and to unit. Then $M = \overline{ee-2be+1} \times \overline{ee-2ce+1} \times \overline{ee-2de+1} = (\text{because } \overline{ee-2ae+1} = 0, \text{ and } \overline{ee+1} = 2ae) \overline{2ae-2be} \times \overline{2ae-2ce} \times \overline{2ae-2de} \times \&c. = e^{n-1} \times \overline{2a-2b} \times \overline{2a-2c} \times \overline{2a-2d} \times \&c.$

In the same manner $N = f^{n-1} \times \overline{2a-2b} \times \overline{2a-2c} \times \overline{2a-2d} \times \&c.$ Therefore if I represent the product of the differences by which $2a$ the middle coefficient of the first divisor exceeds $2b, 2c, 2d, \&c.$ the middle coefficients of the other divisors; then $M = e^{n-1} I$ & $N = f^{n-1} I.$ Therefore, by substituting those expressions for M & N in the values of L and l in the last

$$\text{article } L = \frac{e^{n-r} - f^{n-r}}{I \times \overline{e-f}} \text{ when } n \text{ is greater than } r; \text{ or } L =$$

$$\frac{e^{r-n} - f^{r-n}}{I \times \overline{f-e}} \text{ when } n \text{ is less than } r; \text{ that is the difference of } n$$

$$\text{and } r \text{ being represented by } m, L = \frac{e^m - f^m}{+ I \times \overline{e-f}} \text{ where the sign}$$

of

of I is positive or negative, according as n is greater or less than

r . In like manner $l = \frac{e^{n-r-1} - f^{n-r-1}}{I \times e-f}$ when $n-1$ is greater

than r , or $l = \frac{e^{r-n+1} - f^{r-n+1}}{I \times e-f}$ when $n-1$ is less than

r ; that is $l = \frac{e^{m-1} - f^{m-1}}{I \times e-f}$ in the former case, and $l =$

$\frac{e^{m+1} - f^{m+1}}{I \times e-f}$ in the latter. Because e and f are the roots of the

equation $xx - 2ax + 1 = 0$, $e = a + \sqrt{aa-1}$, $f = a - \sqrt{aa-1}$,

and $e-f = 2\sqrt{aa-1}$, and $L = \frac{a + \sqrt{aa-1}^m - a - \sqrt{aa-1}^m}{+ I \times 2 \sqrt{aa-1}}$

$= \frac{a + \sqrt{aa-1}^m - a - \sqrt{aa-1}^m}{+ 2 I \sqrt{1-aa}} \times \sqrt{-1}$. Hence if Bb , the

FIG. 301.

co-sine of the ark AB be equal to a , the radius OA being unit, the ark AQ be to the ark AB as m to 1 , and Qq be the co-sine of AQ , then Ob being equal to $\sqrt{1-aa}$, it follows (by comparing the value of S determined in art. 760.) that $L =$

$\frac{+ Oq}{I \times Ob}$. Let the ark QZ be made equal to AB so that AZ may

be equal to $\overline{m-1} \times AB$ or $\overline{m+1} \times AB$ according as $n-1$ is greater or less than r , and Zz be the co-sine of the ark AZ ;

then $l = \frac{+ Oz}{Ob \times I}$. Therefore the fraction $\frac{L-lx}{1-2ax+xx} =$

$\frac{+ Oq + Oz \times x}{Ob \times I} \times \frac{I}{1-2ax+xx}$. The values of the other fractions are similar.

And thus it appears how the fluent of

$\frac{x^r}{x^2 - Ax^{2n-1} + Bx^{2n-2} - \&c.}$ is assignable by circular arks and

logarithms when the denominator is the product of any quadratic divisors.

781. If the values of L and l are to be expressed algebraically,

ly, then raise $a + 1$ to the power of the exponent m , the difference of n and r , by art. 748. multiply the 2d, 4th, 6th, &c. terms of this power by 1, $aa - 1$, $aa - 1^2$, $aa - 1^3$, &c. respectively; and the sum of the products divided by $2a - 2b \times 2a - 2c \times 2a - 2d \times \&c.$ will be equal to $+L$ or $-L$ according as n is greater or less than r . The other coefficient l of the fraction $\frac{L - lx}{1 - 2ax + xx}$ is found by multiplying the like terms of the power of $a + 1$ of the exponent $n - r - 1$ or $r - n + 1$ (according as $n - 1$ is greater or less than r) by 1, $aa - 1$, $aa - 1^2$, $aa - 1^3$, &c. respectively, dividing the sum of the products by $2a - 2b \times 2a - 2c \times 2a - 2d \times \&c.$ The quotient will be equal to $+l$ in the former case, but to $-l$ in the latter. The coefficients M and m , N and n , &c. are found in like manner. But when $n = r$, the coefficients L , M , N , &c. vanish; and when $n - 1 = r$, the other coefficients l , m , n , &c. vanish.

782. When the fraction that is to be resolved in this manner is of the form $\frac{1}{1 - 2zx^n + x^{2n}}$, where the denominator is a tri-

nomial, the coefficients of the fractions $\frac{L - lx}{1 - 2ax + xx}$, &c. into which it is to be resolved, may be more briefly determined from art. 779. by which (substituting in this case o for r) $L =$

$\frac{e}{N \times e - f} - \frac{f}{M \times e - f}$; where, P being supposed equal to

$xx - 2ax + 1 \times Q$, M is the value of Q when $x = e$, and N its value when $x = f$. By taking the fluxions, as in that article,

$\dot{P} = \frac{2nx^{2n-1} \cdot x - 2nax^{n-1} \cdot x}{xx - 2ax + 1} = \frac{2xx - 2ax \times Q + \dot{Q} \times xx - 2ax + 1}{xx - 2ax + 1}$, and substituting e for x , and M for Q , $2ne^{2n-1}$

$- 2nxe^{n-1} = \frac{2e - 2a \times M}{e - f} \times M$; consequently $e - f$

$\times M = 2ne^{n-1} \times e^n - z$. In the same manner $f - e \times N =$

$2nf^{n-1} \times f^n - z$. But $e^{2n} - 2ze^n + 1 = 0$ and $e^n + \frac{1}{e^n}$ (or

e^n

$e + f^n) = 2l$, and $e^n - f^n = 2e^n - 2l$ or $2l - 2f^n$; so that $\overline{e-f} \times M = ne^{n-1} \times \overline{e^n - f^n}$ and $\overline{e-f} \times N = nf^{n-1} \times \overline{e^n - f^n}$.

Therefore $L = \frac{e}{nf^{n-1} \times \overline{e^n - f^n}} - \frac{f}{ne^{n-1} \times \overline{e^n - f^n}} = \frac{1}{n}$. By

art. 779. $l = \frac{-1}{M \times \overline{e-f}} - \frac{1}{N \times \overline{f-e}} = \frac{-1}{ne^{n-1} \times \overline{e^n - f^n}}$

$+ \frac{1}{nf^{n-1} \times \overline{e^n - f^n}} = \frac{1}{n} \times \frac{e^{n-1} - f^{n-1}}{e^n - f^n} = \frac{1}{n} \times$

$\frac{a + \sqrt{aa-1}^{n-1} - a - \sqrt{aa-1}^{n-1}}{a + \sqrt{aa-1}^n - a - \sqrt{aa-1}^n}$. Therefore, by comparing

the value of S in art. 760, it appears that if Bb the co-sine of the ark AB be equal to a , $AK = n \times AB$, $AZ = \frac{1}{n-1} \times AB$, Kk and Zz be the co-sines of AK and AZ ; then $l = \frac{1}{n} \times$

$\frac{Oz}{Ok}$; and the fraction $\frac{L-lx}{1-2ax+xx} = \frac{\frac{1}{n} - \frac{Oz}{n \times Ok} \times x}{1-2ax+xx}$, which coincides with Mr. DE MOIVRE's fifth proposition, *lib. I. Miscel. Analyt.* as it is concisely expressed, p. 42. of that treatise.

783. When the fraction proposed is of this form $\frac{x^r}{1-2lx^n+x^{2n}}$

r being any integer positive number less than $2n$; let the difference of n and r be represented by m as above; and $L =$ (art.

780.) $\frac{e^m - f^m}{1 \times \overline{e-f}} =$ (because $1e^{n-1} = M = ne^{n-1} \times \frac{e^n - f^n}{e-f}$

by what was proved in the last article) $\frac{1}{n} \times \frac{e^m - f^m}{e^n - f^n}$. There-

fore if the ark AQ be to AB as m to 1 ; and Qq be the co-sine of

of AQ, $L = \frac{Oq}{n \times Ok}$; and $l = \frac{e^{m-1} - f^{m-1}}{1 \times e - f}$ or $\frac{e^{m+1} - f^{m+1}}{1 \times e - f}$

according as n is greater or less than r ; that is $l = \frac{1}{n} \times$

$\frac{e^{m-1} - f^{m-1}}{e^n - f^n}$ or $\frac{1}{n} \times \frac{e^{m+1} - f^{m+1}}{e^n - f^n}$; or, supposing AZ =

$\overline{m+1} \times AB$, and Zz to be the co-sine of AZ, $l = \frac{Oz}{n \times Ok}$. There-

fore the fraction $\frac{L - lx}{1 - 2ax + xx} = \frac{Oq - Oz \times x}{n \times Ok \times 1 - 2ax + xx}$. When

$r = n$, or to $n - 1$, the numerators of those fractions consist of one term only.

784. It appears, as in article 773. that the fraction $\frac{1 + px + qx^2 + rx^3 + \&c.}{x^n - Ax^{n-1} + Bx^{n-2} - \&c.}$ is equal to $\frac{K}{x-a} + \frac{L}{x-b} + \frac{M}{x-c}$

$+ \frac{N}{x-d} + \&c.$ If a, b, c, d , the roots of the equation x^n

$- Ax^{n-1} + Bx^{n-2} - \&c. = 0$ be all unequal, the index of x in the numerator $1 + px + qx^2 + \&c.$ be less than n , and we suppose the coefficients K, L, M, N, &c. respectively equal to the quantities that result when we substitute successively $a, b,$

$c, d, \&c.$ for x in $\frac{1 + px + qx^2 + rx^3 + \&c.}{nx^{n-1} - n-1 \times Ax^{n-2} + n-2 \times Bx^{n-3} - \&c.}$

This theorem serves for reducing briefly the fluent of $\frac{1 + px + qx^2 + \&c.}{x^n - Ax^{n-1} + Bx^{n-2} - \&c.}$ $\times x$ to logarithms or circular

arks.

785. Suppose now that some of the factors of the denominator of the fraction, by which the ordinate is expressed, are equal to each other; and let it be required to find the fluent or area. For example, let it be required to find the fluent of

$\frac{1 + px + qx^2 + \&c.}{x+a \times x+b} \times x$. Suppose $\frac{1 + px + qx^2 + \&c.}{x+a \times x+b} =$

$\frac{H}{x+a} + \frac{m}{x+b}$

$$\frac{H}{x+a} + \frac{K}{x+a} + \frac{L}{x+a} \dots + \frac{b}{x+b} + \frac{k}{x+b} + \frac{l}{x+b} + \&c. \text{ where } H, K, L, b, k, l, \&c. \text{ are the in-}$$

variable coefficients that are to be determined, and the fractions of each sort are to be continued till their number be equal to m and n respectively. By reducing the equation to a common denominator, $H \times \overline{x+b}^n + K \times \overline{x+a} \times \overline{x+b}^n + L \times \overline{x+a}^2 \times \overline{x+b}^n \dots + b \times \overline{x+a}^m + k \times \overline{x+b} \times \overline{x+a}^m + l \times \overline{x+b}^2 \times \overline{x+a}^m + \&c. = 1 + px + qx^2 + rx^3 + \&c.$ By supposing $x+a=0$, or $x=-a$, all the Terms of this equation that involve $x+a$ vanish, and we find $H \times \overline{b-a}^n = 1 - pa + qa^2 - ra^3 + \&c.$ or $H = \frac{1 - pa + qa^2 - \&c.}{\overline{b-a}^n}$. By ta-

king the fluxion of the equation, dividing each term by $\dot{x}$, and then supposing $x=-a$, we have $nH \times \overline{b-a}^{n-1} + K \times \overline{b-a}^n = p - 2qa + 3ra^2 - \&c.$ By taking the fluxions again, dividing by $\dot{x}$ and supposing $x=-a$, we find $n \times n-1 \times H \times \overline{b-a}^{n-2} + 2nK \times \overline{b-a}^{n-1} + 2L \times \overline{b-a}^n = 2q - 6ra + \&c.$ Thus the values of $H, K, L, \&c.$ are easily computed; and by proceeding in like manner and supposing $x+b=0$, the values of $b, k, l, \&c.$ are determined. If the fraction proposed

be $\frac{1}{\overline{x+a}^m \times \overline{x+b}^n}$, that is, if $p, q, r, \&c.$ be supposed to vanish,

$$\text{then } H = \frac{1}{\overline{b-a}^n}, K = \frac{-n}{\overline{b-a}^{n+1}}, L = n \times \frac{n+1}{2} \times \frac{1}{\overline{b-a}^{n+1}}, b = \frac{1}{\overline{a-b}^m}, k = \frac{-m}{\overline{a-b}^{m+1}}, l = m \times \frac{m+1}{2} \times \frac{1}{\overline{a-b}^{m+1}}, \&c. \text{ which}$$

coincides with Mr. LEIBNITZ's theorem, *Act. Lips.* 1703. If the

the fraction proposed be $\frac{I}{x+a^m \times x+b^n \times x+c^s}$, and we suppose

it equal to $\frac{H}{x+a^m} + \frac{K}{x+a^{m-1}} + \frac{L}{x+a^{m-2}} + \&c.$ it will appear in

the same manner that $H = \frac{I}{b-a \times c-a}$, $K = \frac{-mH}{b-a} - \frac{sH}{c-a}$, —

$2C = \frac{\frac{m \times m-1}{2}}{b-a} + \frac{2ms}{b-a \times c-a} + \frac{\frac{s \times s-1}{2}}{c-a} \times H + \frac{2m}{b-a} - \frac{2s}{c-a}$
 $\times K, \&c.$

786. The ordinate or fraction $\frac{I + px + qx^2 + \&c.}{x+a^m \times x+b^n}$, be-
 ing resolved in this manner, the fluent will be found (by art.
 737.) equal to $\frac{-H}{m-1 \times x+a^{m-1}} + \frac{-K}{m-2 \times x+a^{m-2}} + \dots + \frac{-b}{n-1 \times x+b^{n-1}}$
 $\frac{-k}{n-2 \times x+b^{n-2}} \&c.$ If the last coefficient of each sort vanish,

the fluent will be assignable in algebraic terms. In other cases
 the fluent is assigned by logarithms with algebraic quantities.

787. If we suppose the fraction $\frac{I + px + qx^2 + \&c.}{x+a^m \times x+b^n} =$
 $\frac{K}{x+b^n} + \frac{A+Bx+Cx^2 \dots Gx^{m-1}}{x+a^m}$, that it may be resolved af-

ter Mr. DE MOIVRE's method, *Miscell. Analyt.* p. 59. Then
 $K \times x + a^m + Ab + A + Bb \times x + B + Cb \times x^2 + C + Db$
 $\times x^3 = I + px + qx^2 + \&c.$ By supposing $x + b = 0$, or
 $x = -b$, $K \times a-b^m = I - pb + qb^2 - \&c.$ The supposi-
 tion of $x = 0$ gives $Ka^m + Ab = I$. By taking the fluxions
 of the equation, dividing by x and supposing $x = 0$, nKa^{m-1}
 $+ \dots$

$\pm A \pm Bb = p$; and by proceeding in this manner the coefficients $K, A, B, C, \&c.$ may be determined. If the fraction be $\frac{1}{x+a \times x+b}$, the coefficient of any term in the numerator of the

second fraction, as of Fx^r , is found by raising $a-b$ to the power of the exponent m , rejecting as many of the terms of this power $a^m, ma^{m-1}b, \&c.$ as there are units in $r \pm 1$; and dividing the sum of these that remain by $b^{r \pm 1} \times a-b^m$; for the quotient will give $\pm F$ or $-F$, according as r is an even or odd number. The fraction $\frac{1}{1-ax \times 1-bx}$ is resolved in like

manner by a similar rule, and supposing it equal to $\frac{K}{1-bx} \pm \frac{A+Bx+Cx^2 \dots Gx^{m-1}}{1-ax}$; $K = \frac{b^m}{b-a}, A = 1 - \frac{b^m}{b-a}$

$$B = bA \pm \frac{mab^m}{b-a}, C = bB - \frac{m \times m-1 \times b^m a^2}{2 \times b-a}, \&c.$$

788. The fluxion $\frac{x^r \dot{x}}{e \pm fx^n}$ is transformed into $\frac{rz^{m+r-1} \dot{z}}{e \pm fz^{rn}}$

by supposing $x^{\frac{1}{r}} = z$; because $x^{\frac{m+r}{r}} = z^{m+r}, x^{\frac{m}{r}} \dot{x} = rz^{m+r-1} \dot{z}$ and $x^n = z^{rn}$. In like manner the fluxion is transformed, so as to become rational, when the denominator is a trinomial; and the fluent may be found by the preceeding articles.

789. Sir ISAAC NEWTON has given some excellent theorems for reducing fluents to others of a more simple form in the 7th, 8th and 11th prop. of his treatise *de quadrat. curvar.* Let $R = e + fx^{2n} + gx^{2n} + bx^{3n} + \&c.$ and consequently $\dot{R} = nfx^{2n-1} \dot{x} + 2ngx^{2n-1} \dot{x} + 3nbx^{3n-1} \dot{x} + \&c.$ Let $\dot{A} =$

$x^{m-1} \dot{x} R^l, \dot{B} = \dot{A}x^n, \dot{C} = \dot{B}x^n, \dot{D} = \dot{C}x^n, \&c.$ and $ln + n = p$. Then $meA + p + m \times fB + 2p + m \times gC + 3p + m \times bD + \&c. = x^m R^{l+1}$. This theorem will appear by taking the fluxion of $x^m R^{l+1}$, which (by art. 725.) is $x^{m-1} R^l \times m R \dot{x} + l+1 \times x \dot{R} = x^{m-1} \dot{x} R^l \times me + mfx^n + mgx^{2n} + \&c. + l+1 \times nfx^n + l+1 \times 2ngx^{2n} + \&c. = me\dot{A} + p + m \times f\dot{B} + 2p + m \times g\dot{C} + 3p + m \times b\dot{D} + \&c.$ Let the number of terms in $e + fx^n + gx^{2n} + \&c.$ the value of R be represented by q ; and if as many of the successive areas $A, B, C, D, \&c.$ be known as there are units in $q-1$, the rest can be computed from these, by this theorem. Thus if R be a binomial $e + fx^n$, any one of the areas, $A, B, C, D, \&c.$ being given, the rest may be computed from it; and when R is a trinomial $e + fx^n + gx^{2n}$, any two of those areas, as A and B , are sufficient for determining the rest.

790. Let $\dot{H} = x^{m-1} \dot{x} R^{l+1} = x^{m-1} \dot{x} R^l \times e + fx^n + gx^{2n} + \&c. = e\dot{A} + f\dot{B} + g\dot{C} + b\dot{D} + \&c.$ and it follows that $H = eA + fB + gC + bD + \&c.$ Hence it appears that m and l being any numbers whatsoever, if r and s be any integer numbers, and as many of the areas $A, B, C, D, \&c.$ be known as there are units in $q-1$, the fluent of $x^{m+rn-1} \dot{x} \times e + fx^n + \&c.$ may be computed from them.

791. In like manner it appears that if $R = e + fx^n + gx^{2n} + \&c. S = E + Fx^n + Gx^{2n} + \&c. \dot{A} = x^{m-1} \dot{x} R^l S^k, \dot{B} = \dot{A}x^n, \dot{C} = \dot{B}x^n, \&c.$ and $ln + n = p, kn + n = q$, then $x^m R^{l+1} S^{k+1} = meEA + meF + mfe + pfe + qeF \times B + meG + mgE + mfg + pfg + qfF + 2pgE + 2qgG \times C + \&c.$

792. From these, particular theorems are easily deduced that may

may be of use in the resolution of problems. Let R be the binomial $e - fx^n$; and, as in art. 789. let $\dot{A} = x^{m-1} \dot{x} R^l$, $\dot{B} = \dot{A}x^n$, $\dot{C} = \dot{B}x^n$, $\dot{D} = \dot{C}x^n$, &c. And $M = m + n + m$. Let m and $l+1$ be any positive numbers whatsoever that $x^m R^{l+1}$ may vanish either when $x=0$, or $e - fx^n = 0$; and let r be any integer positive number; then if A represent the fluent of $x^{m-1} \dot{x} \times e - fx^n^l$ that is generated while x flows and from being 0 becomes equal to $\frac{e^{\frac{1}{n}}}{f}$, the fluent of $x^{m+rn-1} \dot{x} \times e - fx^n^l$ generated in the same time will be to $A \times \frac{e^r}{f^r}$ as $\frac{m}{M} \times \frac{m+n}{M+n} \times \frac{m+2n}{M+2n} \times \frac{m+3n}{M+3n} \times \dots$ to 1; where the fractions $\frac{m}{M}$, &c. are to be continued till their number be equal to r . For in the present case $m\dot{C}A - Mf\dot{B} = 0$ by art. 789. and $B = \frac{m}{M} \times \frac{eA}{f}$, $C = \frac{m+n}{M+n} \times \frac{eB}{f} = \frac{m}{M} \times \frac{m+n}{M+n} \times \frac{eeA}{ff}$, $D = \frac{m+2n}{M+2n} \times \frac{eC}{f} = \frac{m}{M} \times \frac{m+n}{M+n} \times \frac{m+2n}{M+2n} \times \frac{e^3A}{f^3}$, and so on. The same theorem serves when $\dot{A} = x^{m-1} \dot{x} \times \overline{fx^n - e}^l$.

793. For example, let $\dot{A} = \frac{\dot{x}}{\sqrt{1-xx}}$, and consequently A is equal to the fourth part of the circumference of the circle whose radius is 1, and the fluent of $\frac{x^{2r} \dot{x}}{\sqrt{1-xx}}$ generated while x flows, and from being 0 becomes equal to 1, will be to A as $\frac{1 \times 3 \times 5 \times 7 \times \dots}{2 \times 4 \times 6 \times 8 \times \dots}$ to 1; where the fractions are to be continued till their number be equal to r . Because in this case $\dot{A}$ or $x^{m-1} \dot{x} \times \overline{fx^n - e}^l = x^{1-1} \dot{x} \times \overline{1 - xx}^{-2}$, so that $m = 1$, $l =$

$l = -\frac{1}{2}, n=2, M=2, e=1$ and $f=1$. The fluxion $\frac{\dot{x}}{\sqrt{1-xx}}$ is transformed into $\frac{\dot{z}}{1+zz}$, by supposing $x = \frac{z}{\sqrt{1+zz}}$, and $\frac{x^{2r} \dot{x}}{\sqrt{1-xx}}$ into $\frac{\dot{z} z^{2r}}{1+zz}$; the fluent of which is therefore to A the fluent of $\frac{z}{1+z^2}$ (or the quadrantal ark of the circle of the radius 1) as $\frac{1 \times 3 \times 5 \times 7}{2 \times 4 \times 6 \times 8} \times \&c.$ to 1. If we suppose $A = \dot{x} \sqrt{1-xx}$, in which case A is the fourth part of the area of the circle whose radius is 1, then the fluent of $x^{2r} \dot{x} \sqrt{1-xx}$ will be to A as $\frac{1 \times 3 \times 5 \times 7}{4 \times 6 \times 8 \times 10} \times \&c.$ to 1. By supposing $x = \frac{z}{\sqrt{1+zz}}$, the corresponding fluent of $\frac{\dot{z} z^{2r}}{1+zz}$ will be to the fluent of $\frac{\dot{z}}{1+zz}$, in the same ratio. In like manner other theorems of this kind may be deduced from those in art. 789, &c.

794. The fluxion $x^{m-1} \dot{x} \times \overline{e+fx^n}^l$ being transformed, as in art. 742. by supposing $e+fx^n = z$, the fluent will be measured by the areas of conic sections when $\frac{m}{n}$ is any integer number positive or negative, by art. 789. When $\frac{m}{n} + l$ is any integer number, the same will appear by supposing $z = \frac{e+fx^n}{x^n}$. Or if l be equal to the fraction $\frac{q}{s}$, we

may suppose $z = \overline{e+fx^n}^{\frac{1}{s}}$ in the former case, and $z = \frac{\overline{e+fx^n}^{\frac{1}{s}}}{x^n}$

in the latter. The fluxion $x^{rn-1} \dot{x} \times \overline{e+fx^n}^l$ is transformed, by

sup-

by supposing $\frac{e + fx^n}{g + bx^n} = z$, (and consequently $x^n = \frac{e - gz}{bz - f}$

and $\frac{\dot{x}}{x} = \frac{gf - eb}{e - gz \times bz - f} \times \dot{z}$) into $\frac{gf - eb}{n} \times z^l \dot{z} \times \frac{e - gz^{r-1}}{bz - f^{r+1}}$.

By supposing $z = gx^n + \frac{1}{2}f$, the fluxion $x^{rn-1} \dot{x} \times \frac{e + fx^n + gx^{2n}}{e - ff + zz} + \frac{1}{2}$ is transformed into $\frac{z}{n} \times \frac{2z - f}{2g}^{r-1} \times \frac{e - ff + zz}{g} + \frac{1}{2}$; and the fluent may be found in both those

cases by the preceeding articles when r is any integer number.

If we suppose $y = \frac{\sqrt{e + fx^n + gx^{2n}} - \sqrt{e}}{x^n}$, and transform

the last fluxion from x to y , its expression will become rational, as is shewn, *Miscel. analyt.* p. 65. When any of those fluxions is multiplied or divided by a rational binomial $E + Fx^n$, or trinomial $E + Fx^n + Gx^{2n}$, or by any quantity that can be resolved into such binomial or trinomial factors, the fluent may be measured by the areas of conic sections (that is either by algebraic quantities, or by circular arks, or logarithms, or these compounded together) by the preceeding articles.

795. When a fluxion is proposed that involves an irrational quantity, the fluent is sometimes obtained in finite terms, or compared with a circular ark or logarithm, by supposing the quantity that is under the radical sign equal to a new flowing

quantity. Thus if $\dot{Q} = \frac{\dot{x}}{E + Fx \times e + fx} \times \frac{E + Fx}{e + fx}^{\frac{m}{n}}$, and we

suppose $z = \frac{E + Fx}{e + fx}$, then $\frac{\dot{z}}{z} = \frac{Fe - Ef}{E + Fx \times e + fx} \times \dot{x}$, and

$\dot{Q} = \frac{z^{\frac{m}{n}-1}}{Fe - Ef}$; consequently the fluent $Q = \frac{nz^{\frac{m}{n}}}{m \times Fe - Ef} =$

$\frac{n}{m \times Fe - Ef} \times \frac{E + Fx}{e + fx}^{\frac{m}{n}}$. But this is often more easily obtain-

ed by transforming the fluxion from the sine or co-sine of an ark to the tangent or secant, or to the sum or difference of the secant and tangent, or by the converse operations. If we suppose

$x = \frac{z}{\sqrt{1+zz}}$ (z being the tangent that corresponds to the

sine x) then $\frac{\dot{x}}{\sqrt{1-xx}} = \frac{\dot{z}}{1+zz}$. Hence if $\dot{Q} = \frac{-\dot{x}}{x^2\sqrt{1-xx}}$

then $\dot{Q} = \frac{-\dot{z}}{z^2}$, and $Q = \frac{1}{z} = \frac{\sqrt{1-xx}}{x}$. And if $\dot{Q} =$

$\frac{aax}{aa+xx \times \sqrt{1-xx}} = \frac{aaz}{aa+aa-1 \times zz}$, then Q is equal to the

ark of a circle described with the radius $\frac{a}{\sqrt{aa+1}}$ that has its

tangent equal to z or $\frac{x}{\sqrt{1-xx}}$. If we suppose $x + \sqrt{xx+1}$

$= z$ then $\frac{\dot{x}}{\sqrt{xx+1}} = \frac{\dot{z}}{z}$. If we suppose $\frac{a + \sqrt{aa+xx}}{x} = z$,

then $\frac{-a\dot{x}}{x\sqrt{aa+xx}} = \frac{\dot{z}}{z}$: So that the fluent is the logarithm of

z , the modulus being 1. And by supposing $\frac{e + \sqrt{ee+ffx^{2n}}}{x^n} = z$,

$\frac{-\dot{x}}{x\sqrt{ee+ffx^{2n}}} = \frac{\dot{z}}{nez}$; so that the fluent is

$\frac{1}{ne} \times \log. z$, the modulus being unit.

796. Supposing, as in art. 789. $R = e + fx^n$, $\dot{A} = x^{m-1}$
 $\dot{x} R^l$, and $\dot{B} = \dot{A}x^n$, we found $meA + MfB = x^m R^{l+1}$;
 from which it follows, that if neither $m = 0$, nor $M = 0$, A and
 B depend mutually upon each other; but if $m = 0$, B is af-
 signable in finite algebraic terms; and if $M = 0$, A is assignable
 in such terms. If neither $\frac{m}{n}$ nor $\frac{M}{n}$ be equal to 0, or to an in-
 teger number, the fluents of all the fluxions in the series,
 $\dot{A}x^{2n}$

$\dot{A}x^{2n}$, $\dot{A}x^n$, $\dot{A}$, $\dot{A}x^{-n}$, $\dot{A}x^{-2n}$, &c. (which may be continued either way) depend upon the fluent of any one fluxion in the series; but when either $\frac{m}{n}$ or $\frac{M}{n}$ is an integer, or when either of them vanishes, this cannot be said of the whole series.

Let $\dot{A} = \frac{ax}{x^2\sqrt{aa-xx}}$, where $M=0$, $m=-1$, and $A = \frac{\sqrt{aa-xx}}{ax}$, but the fluent of $\dot{A}x^2 (= \frac{ax}{\sqrt{aa-xx}})$ is the circular ark whose sine is x , the radius being a : The fluents of

$\dot{A}x^{-2}$, $\dot{A}x^{-4}$, &c. depend upon the former, and are assignable in finite algebraic terms; but the fluents of $\dot{A}x^4$, $\dot{A}x^6$, &c. depend upon the latter, and are assignable by that circular ark

with algebraic quantities. If $\dot{A} = \frac{-xx}{\sqrt{aa-xx}}$, $m=0$, $M=1$,

$A = \sqrt{aa-xx}$, and the fluents of $\dot{A}x^2$, $\dot{A}x^4$, &c. are assignable

by algebraic quantities; but the fluent of $\dot{A}x^{-2} (= \frac{-x}{x\sqrt{aa-xx}})$

is the logarithm of $\frac{a + \sqrt{aa-xx}}{x}$, the modulus being unit, and

the fluents of $\dot{A}x^{-4}$, $\dot{A}x^{-6}$, &c. depend upon this logarithm.

In like manner if $\dot{A} = \frac{x}{\sqrt{xx-1}}$, A is the logarithm of $x + \sqrt{xx-1}$, and the fluents of $\dot{A}x^2$, $\dot{A}x^4$, &c. depend upon it;

but the fluents of $\dot{A}x^{-2}$, $\dot{A}x^{-4}$, &c. are assignable in finite algebraic terms. If $\dot{A} = \frac{xx}{\sqrt{xx-1}}$, $A = \sqrt{xx-aa}$, and the flu-

ents of $\dot{A}x^2$, $\dot{A}x^4$, &c. are assignable in finite algebraic terms;

but the fluent of $\dot{A}x^{-2} (= \frac{x}{x\sqrt{xx-1}})$ is the ark whose secant

is x , the radius being unit, and the fluents of $\dot{A}x^{-4}$, $\dot{A}x^{-6}$, &c.

depend

depend upon it. If $\dot{A} = \frac{x^{m-1} \dot{x}}{\sqrt{+aa+xx}}$, and m be a fraction, the fluents of all the fluxions in the series $\dot{A}x^{-2}$, $\dot{A}x^{-4}$, $\dot{A}x^{-6}$, &c. depend upon A .

797. Let $R = fx^n - e$, $\dot{A} = x^{m-1} \dot{x} R^l$, $\dot{B} = \dot{A}x^{-n}$, $\dot{C} = \dot{B}x^{-n} = \dot{A}x^{-2n}$, $\dot{D} = \dot{C}x^{-n} = \dot{A}x^{-3n}$, &c. and $M = ln + n + m$, as formerly; then when $fx^n = e$, $B = \frac{M-n}{m-n} \times f \frac{A}{e}$, $C = \frac{M-2n}{m-2n} \times f \frac{B}{e}$, $D = \frac{M-3n}{m-3n} \times f \frac{C}{e}$ &c. Therefore r being any integer positive number, if $\dot{Q} = \dot{A}x^{-rn}$, $Q : \frac{Af^r}{e^r} :: \frac{M-n}{m-n} \times \frac{M-2n}{m-2n} \times \frac{M-3n}{m-3n} \times \dots$ (where these fractions are to be continued till their number be equal to r) : 1. For example, let $\dot{A} = \frac{\dot{x}}{x\sqrt{xx-1}}$, $\dot{Q} = \frac{\dot{x}}{x^{2r+1}\sqrt{xx-1}}$, then $Q : A :: \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \frac{7}{8} \times \dots$: 1. these fluents are generated while $\frac{1}{x}$ from being 0 becomes equal to 1.

798. After the fluents that can be accurately assigned in finite terms by common algebraic expressions, and those which can be reduced to circular arks and logarithms, the fluents that deserve the next place are such as are assigned by hyperbolic and elliptic arks; which with the former are all comprehended under these which are measured by the lines that bound the conic sections, (the triangle and circle being figures of this kind,) as the first two are measured by the areas of conic sections. The fluent of

$\frac{\dot{x}}{\sqrt{1+x}}$ is of the first class, that of $\frac{\dot{x}}{\sqrt{x} \times \sqrt{1+x}}$ or of $\frac{\dot{x}}{\sqrt{1+xx}}$

is of the second; but the fluents of $\frac{\dot{x}\sqrt{x}}{\sqrt{1+xx}}$, $\frac{\dot{x}}{\sqrt{x} \times \sqrt{1+xx}}$,

$\frac{\dot{x}}{1+xx} \frac{1}{2}$ and $\frac{\dot{x}}{1+xx} \frac{1}{3}$ are of the third class, and (as far as has

appeared hitherto) cannot be reduced to the former. The fluents of this class are sometimes required in the resolution of useful problems, and our design obliges us to give some account of them likewise.

799. Let AEH be an equilateral hyperbola, that has its cen-
ter in S and vertex in A, AD a right line perpendicular to SA,
suppose $SA = 1$, $SN = x$, and let a circle described with the
radius SN from the center S meet AD in M, let SE bisect the
angle ASM, and meet the hyperbola in E, then the hyperbolic

FIG. 305.

ark AE shall be equal to the fluent of $\frac{x\sqrt{x}}{2\sqrt{xx-1}}$. For let the

ark $AE = s$, $SE = r$, and SP be perpendicular on EP the tan-
gent of the hyperbola in P; then the triangles SMA and SEP
will be similar by art. 181, and $s : r :: SE : EP :: SM :$
 $AM :: x : \sqrt{xx-1}$; but SA, SE and SM are in continued pro-
portion, or $r = \sqrt{x}$, so that $r : x :: 1 : 2\sqrt{x}$; conse-

quently $\dot{s} = \frac{x\sqrt{x}}{2\sqrt{xx-1}}$; and supposing the fluent of $\frac{x\sqrt{x}}{\sqrt{xx-1}}$

to begin to be generated when $x = 1$, and thereafter to in-
crease while x increases, it will be always equal to $2AE$. If
 Am be perpendicular to SM in m , and we now suppose $Sm = x$,

then the hyperbolic ark AE will be the fluent of $\frac{-x}{2x\sqrt{x} \times \sqrt{1-xx}}$

(as will appear by substituting in the former fluxion x^{-1} for x ;) and $EP - AE$ the excess of the tangent above the hyperbolic

ark AE will be the fluent of $\frac{-x\sqrt{x}}{2\sqrt{1-xx}}$; because EP will then

be equal to $\sqrt{\frac{1}{x} - x}$, and its fluxion to $\frac{-x - xxx}{2x\sqrt{x} \times \sqrt{1-xx}}$.

800. Let AB be perpendicular from A the vertex of the hy-
perbola to the asymptote SB in B. Suppose now $SB = 1$, up-
on BA take $BL = x$, join SL and let it meet the hyperbola in
E; from the center S describe the ark AQ, intersecting SE in
Q; and the hyperbolic ark AE shall be equal to the fluent
of

of $\frac{\dot{x} \sqrt{1+xx}}{2x \sqrt{x}}$; because if Ab , LZ and EK be perpendicular to the other asymptote in b , Z and K respectively, Sb . $SK :: EK : Ab (= LZ) :: SK : SZ$, $SZ = BL = x$, $SK = \sqrt{Sb \times Sz} = \sqrt{x}$, $SE^2 = SK^2 + EK^2 = x + \frac{1}{x}$; and the fluxion of AE being to the fluxion of SK as SE to SK , it is therefore equal to $\frac{\dot{x}}{2x} \times \sqrt{x + \frac{1}{x}}$ or $\frac{\dot{x} \sqrt{1+xx}}{2x \sqrt{x}}$. The fluxion of SE or of QE , is

$\frac{xxx - \dot{x}}{2x \sqrt{x} \times \sqrt{xx + 1}}$, by adding which to the fluxion of AE , it appears

that $AE + EQ$ is the fluent of $\frac{\dot{x} \sqrt{x}}{\sqrt{1+xx}}$ which begins to be generated when $x = 1$ (or when $BL = BA$) and thereafter increases while x increases. In the same manner $AE - EQ$ is the fluent of $\frac{-\dot{x} \sqrt{x}}{\sqrt{1+xx}}$ that begins to be generated when $x = 1$, and thereafter increases while x decreases.

801. Suppose $SA = 1$, $AM = x$, and $2AE$ will be the fluent of $\frac{x}{1+xx}^{\frac{1}{4}}$ that vanishes with x ; as appears by substituting in the first value of $\dot{s}$ in art. 799. $\sqrt{1+xx}$ in the place of x . Suppose $SA = 1$, $Am = x$, and $2EP - 2AE$ will be the fluent of $\frac{-x}{1-xx}^{\frac{1}{4}}$ that begins to be generated when $x = 1$, and thereafter increases while x decreases. If we suppose $SB = 1$, $SL = x$, then $AE \mp EQ$ will be the fluent of $\frac{x}{xx-1}^{\frac{1}{4}}$ that begins to be generated when $xx = 2$.

802. As for the fluent of $\frac{\dot{x}}{\sqrt{x} \times \sqrt{1+xx}}$ or of $\frac{\dot{x}}{1+xx}^{\frac{3}{4}}$, it does not appear that it is possible to represent it by any hyperbolic arch and algebraic quantities. But by assuming an elliptic ark likewise, it may be assigned by the following construction. The rest remaining as in art. 799. Let an ellipse ARD be described having its center in S , SF the distance of the focus F from

from the center S equal to the shorter semi-axis SA and consequently the semi-transverse axis $SD : SA :: \sqrt{2} : 1$. Suppose $SA = 1$, $Sm = x$, take SX upon SA equal to SP (or to a mean proportional betwixt SA and Sm,) let the ordinate XR meet the ellipse in R; and the fluent of $\frac{-\dot{x}}{2\sqrt{x} \times \sqrt{1-xx}}$ that begins to be generated when $x = 1$, and thereafter increases while x decreases, will be equal to $AR + AE - EP$, the difference by which the sum of the elliptic and hyperbolic arcs AR and AE exceeds EP the tangent of the latter. For $SX = \sqrt{x}$, and if RT the tangent of the ellipse at R meet SA in T, $ST = \frac{1}{\sqrt{x}}$, $XT = ST - SX = \frac{1-x}{\sqrt{x}}$, $XR^2 = 2 \times \frac{1-x}{\sqrt{x}}$, $RT^2 = XT^2 + XR^2 = \frac{1-xx}{x}$, and the fluxion of the elliptic ark AR will be to the fluxion of SX as RT to XT, that is as $\sqrt{1-xx}$ to $1-x$, or as $1+x$ to $\sqrt{1-xx}$; consequently (the fluxion of SX being $\frac{\dot{x}}{2\sqrt{x}}$) the fluxion of the ark AR is $\frac{-\dot{x}}{2\sqrt{x}} \times \frac{1+x}{\sqrt{1-xx}} = \frac{-\dot{x}}{2\sqrt{x} \times \sqrt{1-xx}} - \frac{\dot{x}\sqrt{x}}{2\sqrt{1-xx}}$; and the fluent of $\frac{-\dot{x}}{2\sqrt{x} \times \sqrt{1-xx}}$ (by the latter part of art. 799.) equal to $AR + AE - EP$. If we suppose $An = x$, $AR + AE - EP$ will be the fluent of $\frac{\dot{x}}{2 \times \frac{1-xx}{\sqrt{1-xx}} |^{\frac{3}{4}}}$; as will appear by substituting $\sqrt{1-xx}$ for x in the former fluxion. By supposing $BL = z$, and $SB = 1$, the same difference $AR + AE - EP$ gives the fluent of $\frac{\dot{z}}{\sqrt{z} \times \sqrt{1+zz}}$; because if $Sn = x$, then $x = \frac{2\sqrt{z \times z}}{1+zz}$. It is likewise the fluent of $\frac{\dot{z}}{zz-1 |^{\frac{3}{4}}}$, if we suppose $SL = z$ and $SB = 1$, or of $\frac{\dot{z}}{2 \times \frac{1+zz}{\sqrt{1+zz}} |^{\frac{3}{4}}}$ if we suppose $AM = z$, and $SA = 1$.

803. The fluent of $\frac{-x}{2\sqrt{x} \times \sqrt{1-xx}}$ (which we found equal to $AR + AE - EP$, art. 802.) is equal to AP the ark of the curve that is the *locus* of P , (where the perpendiculars from the center S intersect the tangents of the equilateral hyperbola) which is called the *lemniscata*. For (art. 212.) the fluxion of the curve AP is to the fluxion of SP as SE to EP , or as SA to Am , that is (supposing $SA = 1$, and $Sm = x$) as 1 to $\sqrt{1-xx}$; but $SP : SA :: SA : SE$, and $SE = \frac{1}{\sqrt{x}}$; consequently $SP = \sqrt{x}$, the

fluxion of SP is $\frac{x}{2\sqrt{x}}$ and the fluxion of $AP = \frac{-x}{2\sqrt{x} \times \sqrt{1-xx}}$.

FIG. 306. If F be the focus of the hyperbola, FH perpendicular to the tangent EP in H , then it is known that H will be always found in a circle described from the center S with the radius SA ; and if FH produced meet this circle in b , SP will be equal to $\frac{1}{2} Hb$. From which it appears that the *lemniscata* may be

FIG. 307. constructed in the following easy manner. Bisect SF in f , from the center f describe a circle with a radius equal to $\frac{1}{2} SA$, let any right line SX meet this circle in X and x , set off SP from S on the same right line always equal to the chord Xx and the point P shall be in the *lemniscata*: And the fluents that were described in art. 802. may either be represented by $AR + AE - EP$, or by the ark AP of this curve which is so easily constructed.

804. Let AEH be any other hyperbola, SA the semi-transverse and SD the semi-conjugate axis, $SA = a$, $SD = b$, $e = \frac{aa-bb}{2a}$, $SE = r$, $SP = p$, $AE = s$ and $x = \frac{abb}{pp}$, then the hyper-

bolic ark AE shall be equal to the fluent of $\frac{x \sqrt{xa}}{2\sqrt{xx+2ex-bb}}$.

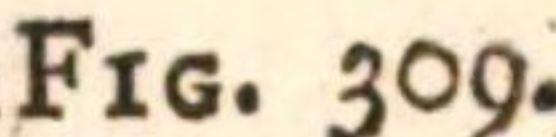
For let SH be the semidiameter conjugate to SE , then $SE^2 - SH^2 = aa - bb = 2ea$, or $SH^2 = rr - 2ea$; and $SH \times SP = SA \times SB = ab$, and $rr - 2ea = \frac{aabb}{pp} = ax$. But $s : r :: SE :$

$$SE : EP :: r : \sqrt{rr - pp} \text{ and } \dot{s} = \frac{rr}{\sqrt{rr - pp}} = \frac{\dot{x}\sqrt{xa}}{2\sqrt{xx + 2ex - bb}}$$

Because $EP = \sqrt{rr - pp} = \sqrt{\frac{axx + 2eax - bb}{x}}$, the fluxion of

EP is $\frac{\sqrt{a}}{2x\sqrt{x}} \times \frac{xxx + bbx}{\sqrt{xx + 2ex - bb}}$, and EP - AE (the excess of the tangent above the hyperbolic ark) is the fluent of $\frac{bbx\sqrt{a}}{2x\sqrt{x} \times \sqrt{xx + 2ex - bb}} - z\sqrt{az}$, or (supposing $z = \frac{pp}{a} = \frac{bb}{x}$) of $\frac{-a^2b^2\dot{p}}{pp\sqrt{a^2b^2 + 2aep^2 - p^4}}$. It appears likewise that the ark AE is the flu-

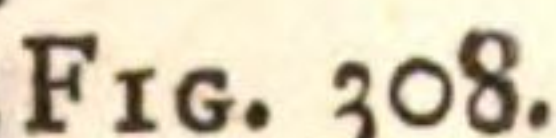
ent of $\frac{-a^2b^2\dot{p}}{pp\sqrt{a^2b^2 + 2aep^2 - p^4}}$, and that EP - AE is the fluent of

$\frac{-p^2\dot{p}}{\sqrt{a^2b^2 + 2aep^2 - p^4}}$. In like manner it appears that if AEB be an  Fig. 309. ellipsis, S the center, SA = a, SB = b, aa + bb = 2ea, SP be perpendicular on the tangent EP in P and SP = p, $x = \frac{abb}{pp}$,

then the ark AE will be the fluent of $\frac{\dot{x}\sqrt{ax}}{\sqrt{2ex - xx - bb}}$ or of

$\frac{-a^2b^2\dot{p}}{pp\sqrt{2aep^2 - a^2b^2 - p^4}}$ that begins to be generated when $p = a$.

805. In order to represent the fluent of $\frac{\dot{z}}{\sqrt{z} \times \sqrt{b^2 + 2ex - zz}}$

or of $\frac{\dot{p}}{\sqrt{a^2b^2 + 2aep^2 - p^4}}$, we must have recourse to both the hyperbolic and elliptic arks. The rest remaining as in the last article, join AD, and let AF perpendicular to AD meet DS  Fig. 308. produced in F, describe an ellipse ARb that has its focus in F, center in S, and SA for the second semi-axis; upon SA take SQ equal to SP, let QR the ordinate at Q meet the ellipse in R

and $\frac{b}{a} \times AR + AE - EP$ shall be the fluent of $\frac{-bb\dot{p}}{\sqrt{aa - pp} \times \sqrt{bb + pp}}$

O o o o

or

or (supposing $pp = az$, and $2ea = bb - aa$ as above) of $\frac{-bbz}{2\sqrt{az} \times \sqrt{bb-2ez-zz}}$. For if R.T the tangent of the ellipse at R meet SA in T, AR = f , SQ (= SP) = p and SF = f , then $ST = \frac{aa}{p}$, $QT = \frac{aa-pp}{p}$, $RT = \frac{\sqrt{aa-pp}}{p} \times \sqrt{aa + \frac{k^2 p^2}{a^2}}$, $\dot{f} : -\dot{p} :: RT : QT$, and $\dot{f} = \frac{-\dot{p}}{a} \times \frac{a^4 + k^2 p^2}{\sqrt{aa-pp} \times \sqrt{a^2 + k^2 p^2}} =$
 (because $kb = aa$ by the supposition) $\frac{-a\dot{p}}{b} \times \frac{bb + pp}{\sqrt{aa-pp} \times \sqrt{bb+pp}}$. But the fluxion of EP - AE was found (art. 802.) equal to $\frac{-ppp}{\sqrt{aa-pp} \times \sqrt{bb+pp}}$. Therefore $\frac{b}{a} \times AR + AE - EP$ is the fluent of $\frac{-bb\dot{p}}{\sqrt{aa-pp} \times \sqrt{bb+pp}}$, or of $\frac{-bbz}{2\sqrt{az} \times \sqrt{bb-2ez-zz}}$ that begins to be generated when p and z are equal to a ; and that the fluent is finite which is generated while z decreases till it vanish, appears from art. 327.

806. The values of those fluents as of $\frac{\dot{p} \sqrt{a^4 + k^2 p^2}}{a \sqrt{aa-pp}}$ (the fluxion of the elliptic ark BR) may be computed either by resolving $\sqrt{\frac{a^4 + k^2 p^2}{aa-pp}}$ into a series, multiplying each term by $\frac{\dot{p}}{a}$, and finding the fluents of the several products by art. 737. which will form a series of algebraic quantities. Or we may compare it with the fluent of $\frac{a\dot{p}}{\sqrt{aa-pp}}$, (the circular ark of the radius a and sine p) by resolving $\sqrt{\frac{a^4 + k^2 p^2}{aa-pp}}$ only into a series, by art 793. Thus $\dot{f} = \frac{a\dot{p}}{\sqrt{aa-pp}} \times 1 + \frac{k^2 p^2}{2a^4} - \frac{k^4 p^4}{8a^8} + \&c.$ consequently the elliptick quadrant ARB is to the quadrant of the circle of the radius a as $1 + \frac{k^2}{4a^2} - \frac{3k^4}{64a^4} + \frac{5k^6}{256a^6} - \&c.$

to 1. The same elliptic quadrant is to the quadrant of the circle of a radius equal to SB the semi-transverse axis, supposing $SB = a$, as $1 - \frac{k^2}{4a^2} - \frac{3k^4}{64a^4} - \frac{5k^6}{256a^6} - \&c.$ to 1.

807. Let $\frac{-bb\dot{p}}{\sqrt{aa-pp} \times \sqrt{bb+pp}} = \dot{Q}$, $\frac{-a\dot{p}}{\sqrt{aa-pp}} = \dot{A}$, then $\dot{Q}$
 $= \frac{-b\dot{p}}{\sqrt{aa-pp}} \times 1 + \frac{pp}{bb} \Big|^{-\frac{1}{2}} = \frac{\dot{A}b}{a} \times 1 - \frac{pp}{2bb} + \frac{3p^4}{8b^4} - \frac{5p^6}{16b^6} + \&c.$

and by art. 793. the fluent Q that is generated betwixt the terms, when $p = 0$ and $p = a$ is (N being supposed to represent the ratio of the semi-circumference of a circle to its diameter)

$Nb \times 1 - \frac{a^2}{4b^2} + \frac{9a^4}{64b^4} - \frac{25a^6}{256b^6} + \&c.$ where the numerical coefficients 1, $\frac{1}{4}$, $\frac{9}{64}$, $\&c.$ are the squares of the several

unciae of a binomial raised to the power of the exponent $-\frac{1}{2}$. If we suppose $x = a - \frac{pp}{a}$, and $E = a + \frac{bb}{a}$, then $\dot{Q} =$

$\frac{bb}{\sqrt{Ea}} \times \frac{\dot{x}}{2\sqrt{ax-xx}} \times 1 - \frac{x}{E} \Big|^{-\frac{1}{2}}$ or $\frac{bb}{a} \times \frac{\dot{x}}{2\sqrt{Ex-xx}} \times 1 - \frac{x}{a} \Big|^{-\frac{1}{2}}$.

Let A first denote the ark of a circle describ'd upon the diameter a , whose vers'd sine is equal to x , and $\dot{A} = \frac{ax}{2\sqrt{ax-xx}}$

$= \frac{1}{2} axx^{-\frac{1}{2}} \times a - x^{-\frac{1}{2}}$; whence $m-1 = -\frac{1}{2}$, $m = \frac{1}{2}$, $n=1$, $l = -\frac{1}{2}$, $e=a$, $f=1$, $M = ln + n + m = 1$; and by art.

792. when r is any integer positive number, the fluent of $\dot{A} \times \frac{x^r}{ar}$ that is generated while x increases from 0 till it become equal to a is $A \times \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \frac{7}{8} \times \&c.$ these fractions being continued till their number be equal to r , and A being supposed now to represent the semi-circumference on the diameter

a . Therefore since $\dot{Q} = \frac{bb\dot{A}}{a\sqrt{aE}} \times 1 + \frac{x}{2E} + \frac{3xx}{8EE} + \frac{5x^3}{16E^3} + \&c.$

It follows that $Q = \frac{Nbb}{\sqrt{aE}} \times 1 + \frac{a}{4E} + \frac{9a^2}{64E^2} + \frac{25a^3}{256E^3} + \&c.$

In like manner $\dot{Q} = \frac{bb}{a} \times \frac{\dot{x}}{2\sqrt{Ex-xx}} \times \sqrt{1-\frac{x}{a}}^{-\frac{1}{2}}$. Whence Q may be compared with an ark of a circle upon the diameter E that has its vers'd sine equal to x .

808. Let $\dot{Q} = \frac{-p^2 \dot{p}}{\sqrt{aa-pp} \times \sqrt{bb+pp}}$ and $\dot{A} = \frac{-a\dot{p}}{\sqrt{aa-pp}}$, then $\dot{Q} = \frac{\dot{A}p^2}{ab} \times \sqrt{1+\frac{pp}{aa}}^{-\frac{1}{2}} = \frac{\dot{A}}{ab} \times p^2 - \frac{p^4}{2b^2} + \frac{3p^6}{8b^4} - \&c.$ Therefore the fluent Q generated betwixt the terms when $p=0$ and $p=a$ is $\frac{Naa}{b} \times \frac{1}{2} - \frac{3a^2}{16b^2} + \frac{15a^4}{128b^4} - \frac{175a^6}{8 \times 256b^6} + \&c.$

This fluent is the ultimate excess of the tangent EP above the hyperbolic ark AE , that is, the limit of this excess while the figure is produced, or (according to the usual manner of expression) the excess of the asymptote above the curve AE when both are supposed to be infinitely produced. By supposing $aa-pp$

$=ax$, $\dot{Q} = \frac{\dot{x}\sqrt{a} \times \sqrt{ax-xx}}{2x\sqrt{E-x}}$, and the same fluent will be found

(by art. 792.) equal to $\frac{Na}{2} \times \sqrt{\frac{a}{E}} + \frac{aA}{2 \times 4E} + \frac{9aB}{4 \times 6E} + \frac{25aC}{9 \times 6E} + \frac{49aD}{8 \times 10E} + \&c.$ where A denotes in the usual manner the first term $\frac{Na}{2} \times \sqrt{\frac{a}{E}}$, B the second term $\frac{aA}{8E}$, C the third term, and so on.

809. It follows from what was shown above, art. 792. 799. &c. that when r is any integer number, the fluent of

$\frac{z^{\frac{rn}{4}-1} \dot{z}}{\sqrt{e+fz^n}}$ is assignable by the arks of conic sections; that is by right lines when r is equal to 4 or to any multiple of 4; by circular and parabolic arks (which may be reduced to logarithms) with right lines, when r is any other even number; by arks of an equilateral hyperbola with right lines, when r is any number of the series 3, 7, 11, 15, &c. and by arks of the same

same hyperbola and right lines, with arks of an ellipsis that has its excentricity equal to the second axis, when r is any of the numbers 1, 5, 9, 13, &c. For if we suppose $z^n = xx$, the pro-

posed fluxion will be transformed into $\frac{2x^{\frac{r}{2}-1}\dot{x}}{n\sqrt{e+fxx}}$; when $r=3$, $\frac{r}{2}-1 = \frac{1}{2}$, and the fluent is found by art. 799 or 800; but when $r=1$, $\frac{r}{2}-1 = -\frac{1}{2}$ and the fluent is found by art. 802.

810. Let n be any fraction whatsoever, and the fluent of FIG. 310.

$\frac{x^n \dot{x}}{\sqrt{xx-1}}$ or $\frac{z^{n-1} \dot{z}}{\sqrt{1-zz}}$ be required. For this end let AL be one

of the figures constructed in art. 392. where the point S, and right line AE were supposed to be given in position, SA was perpendicular to AE in A, M any point upon AE; and the ratio of the angle ASL to ASM, and that of the logarithm of the ray SL to the logarithm of SM, was always the same invariable ratio of n to 1; that is, $ASL : ASM :: n : 1$, and SL to SA as $SM^{\mp n}$ to $SA^{\mp n}$. Let $SA=1$, $SM=x$, $SL=r$, and the ark $AL=s$; consequently $r=x^{\mp n}$, $\dot{r}=\mp nx^{\mp n-1}\dot{x}$, and (by art. 392) $s : \mp \dot{r} :: SM : AM :: x : \sqrt{xx-1}$, or $s =$

$\frac{nx^{\mp n}\dot{x}}{\sqrt{xx-1}}$. Therefore the fluent of $\frac{x^{\mp n}\dot{x}}{\sqrt{xx-1}}$ is $\frac{1}{n} \times s = \frac{1}{n} \times$

AL. If Am be perpendicular to SM in m, then $Sm = \frac{1}{x}$; consequently if we suppose $Sm = z$, then the fluent of

$\frac{-z^{\mp n-1}\dot{z}}{\sqrt{1-zz}}$ will be equal to $\frac{1}{n} \times$ AL. By supposing $z =$

$\sqrt{1-yy}$ and $n = 2-2k$, $\frac{y}{1-yy^{1/k}}$ is transform'd into $\frac{-z^{n-1}\dot{z}}{\sqrt{1-zz}}$,

and the fluent is $\frac{1}{n} \times$ AL. By supposing $y^m = z^2$, $\frac{z^{\frac{nm}{2}-1}\dot{z}}{\sqrt{1-z^m}}$

is transformed into $\frac{2z^{n-1}\dot{z}}{m\sqrt{1-zz}}$, and the fluent is $\frac{2}{mn} \times$ AL.

These

These are the figures which we found to resolve the most simple cases of problems of various kinds in the first book, art. 436. 569, &c.

- FIG. 311. 811. Let Al , Ap , AL , AP , &c. be such a series of figures as was described in art. 212, where each curve is supposed to be always defined by the intersections of the tangents of the preceeding curve with the respective perpendiculars on those tangents drawn from the given point S . Let AL be a figure of the kind described in the last article; that is, let the angle ASL be to ASM , and $\text{Log. } SL$ to $\text{Log. } SM$ always in the same invariable ratio. Then Al , Ap , AP , and all the other figures in the series shall be likewise of this kind. By art. 212. the angle $ASl = ASL \mp 2ASM$, $Sl = x^{\mp n \mp 2}$, and the fluxion of Al to the fluxion of AL as the fluxion of Sl to the fluxion of SL , or as $\frac{n \mp 2}{n} \times x^{\mp 2}$ to n ; consequently the fluxion of AL is $\frac{n \mp 2}{n} \times s x^{\mp 2}$; and the ark Al is assignable by s and algebraic quantities by art. 792. The same is to be said of all the other arks in the series taken alternately, that is, of the 2d, 4th, 6th, &c. from AL . The other curves in the series Ap , AP , &c. are all assignable by AP and right lines; but the arks of any two figures that immediately succeed each other in the series, as of AP and AL , cannot be compared with each other by an algebraic equation. When ApS is supposed to be a semicircle upon the diameter SA , l coincides with A and Al vanishes, AL and the subsequent arks in the series taken alternately (which have all a *cuspid* in S) are assignable by right lines; but the other arks in the series are measured by the circular ark Ap and right lines. When AL is supposed to coincide with the right line AM itself (or $n=1$), P coincides with A , and AP vanishes, Ap is a common parabola that has its focus in S , Al and the arks in the series continued backwards taken alternately from Al admit of a perfect rectification; but the other arks in the same series are measured by parabolic arks and right lines. Of all the figures wherein the angle ASL is to the angle ASM and $\text{log. } SL$ to $\text{log. } SM$ in the same invariable ratio, there are none besides these that seem to admit

admit of a perfect rectification, or an accurate mensuration by circular arks or logarithms. When AL is an equilateral hyperbola that has its center in S (or $n = \frac{1}{2}$) the curves taken alternately from AL either way in the series, are measured by AL and right lines; but the other curves in the series are measured by AL with an elliptic ark (described above art. 802.) and right lines. By supposing $n = \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \&c.$ other series of curves will be formed. And every series of such curves gives two distinct sorts of fluents, which cannot be compared with each other, or with those of any different series of this kind.

C H A P. IV.

Of the area when the ordinate and base are express'd by fluents; of computing fluents from the sums of progressions, or the sums of progressions from fluents, and other branches of this method.

812. **T**HE base being represented by z , and the ordinate of the figure by y , the fluxion of the area is zy . If y and z be both assigned by quantities compounded in common algebraic terms from the powers of the same variable quantity x , the fluxion of the area will be expressed by such quantities multiplied by x . Having insisted on the fluents of such expressions in the preceeding chapters, we now proceed to enquire into the area or fluent when the ordinate is itself assigned by an area or fluent, or when the ordinate and base are both expressed by fluents: And in this case it will be sufficient if we can reduce the area of the figure to the fluents of the former kind; as to circular arks and logarithms, or to elliptic and hyperbolic arks, or, in general, to the fluents of expressions that involve one variable quantity only in algebraic terms with its fluxion. In this case we shall find that the total area (or that which insists upon a certain given base) may be sometimes measured by circular arks or logarithms, though it may not appear that

FIG. 312.

that in the same instances the part of the area can be assigned in this manner which stands upon any segment of the base that may be proposed. For example, let ADa , BEb be concentric circles described from the same center, CB being less than CA ; let AG be the tangent at A , and T any point upon AG ; join CT intersecting the circle BEb in V and v . Now let the figure $CHKR$ be constructed so that the base CR may be always equal to the logarithm of the ratio of $CT \div AT$ to CA , and the ordinate RK always equal to the logarithm of the ratio of Tv to TV , the *modulus* being CA . Then the whole area $CHKLLRC$ generated by the ordinate RK , while the point V describes the quadrant BVE , shall be equal to the rectangle contained by the quadrant AFD and the ark DF whose sine is CB ; but it does not appear that the part of this area $CHKR$, that stands upon any given base CR , can be measured in this manner. The fluents of this kind are sometimes required in the resolution of useful problems, and the mensuration of the whole area is commonly what is most valuable. But before we treat of the area, when the ordinate and base are both express'd by fluents, some theorems are to be premised concerning the area, when the ordinate only is express'd in this manner.

813. Let A represent any area on the base x , suppose $A\dot{x} = \dot{K}$, $K\dot{x} = \dot{L}$, $L\dot{x} = \dot{M}$, $M\dot{x} = \dot{N}$, &c. where K represents the area when the ordinate is A , L the area when the ordinate is K , M the area when the ordinate is L , and so on. Let $x\dot{A} = \dot{B}$, $x\dot{B} = \dot{C}$, $x\dot{C} = \dot{D}$, $x\dot{D} = \dot{E}$, &c. and suppose $B\dot{x} = \dot{k}$, $k\dot{x} = \dot{l}$, $l\dot{x} = \dot{m}$, $m\dot{x} = \dot{n}$, &c. Then shall $K = xA - B$, $2L = xK - k$, $3M = xL - l$, $4N = xM - m$, and so on. For since $A\dot{x} + x\dot{A} = \dot{K} + \dot{B}$, it follows by finding the fluents (art. 738.) that $Ax = K + B$, and $K = Ax - B$. Because $K\dot{x} + x\dot{K} - \dot{k} = \dot{L} + Ax\dot{x} - B\dot{x} = \dot{L} + \overline{Ax - B} \times \dot{x} = \dot{L} + K\dot{x} = 2\dot{L}$, by taking the fluents $xK - k = 2L$. In like manner $L\dot{x} + x\dot{L} - \dot{l} = \dot{M} + Kx\dot{x} - k\dot{x} = \dot{M} + 2L\dot{x} = 3\dot{M}$, and $3M = xL - l$; $M\dot{x} + x\dot{M} - \dot{m} = \dot{N} + Lx\dot{x} - l\dot{x} = \dot{N} + 3M\dot{x} = 4\dot{N}$, and $4N = xM - m$, and so on.

815.

814. In the same manner that $K = xA - B$, it is manifest that $k = xB - C$; consequently $2L = xK - k = x^2A - xB - xB + C = x^2A - 2xB + C$, and $2l = x^2B - 2xC + D$. Hence $6M =$ (by the last art.) $2xL - 2l = x \times x^2A - 2xB + C - x^2B - 2xC + D = x^3A - 3x^2B + 3xC - D$, and $6m = x^3B - 3x^2C + 3xD - E$; $24N = 6xM - 6m = x \times x^3A - 3x^2B + 3xC - D - x^3B - 3x^2C + 3xD - E = x^4A - 4x^3B + 6x^2C - 4xD + E$. And in this manner it is manifest, that if r denote the place of any fluent Z in the series

$$K, L, M, N, \&c. Z = \frac{x^r A - rx^{r-1}B + r \times \frac{r-1}{2} x^{r-2}C - \&c.}{1 \times 2 \times 3 \times 4 \times \dots \times r}$$

which is the first part of *prop. 11. de quadrat. curvar.* When

$$x = a, \text{ then } Z = \frac{a^r A - ra^{r-1}B + r \times \frac{r-1}{2} a^{r-2}C - \&c.}{1 \times 2 \times 3 \dots \times r}$$

$$815. \text{ Let } z = \frac{a^r A - ra^{r-1}B + r \times \frac{r-1}{2} a^{r-2}C - \&c.}{1 \times 2 \times 3 \dots \times r} = \frac{a^r \dot{A} - ra^{r-1} \dot{B} + r \times \frac{r-1}{2} a^{r-2} \dot{C} - \&c.}{1 \times 2 \times 3 \dots \times r}$$

$$\text{consequently } z = \frac{a^r A - ra^{r-1}B + r \times \frac{r-1}{2} a^{r-2}C - \&c.}{1 \times 2 \times 3 \dots \times r}$$

and when $x = a$, $Z = \frac{z}{1 \times 2 \times 3 \times \dots \times r}$; which is the second part of the same proposition.

816. Let $x\dot{A} = \dot{P}$, $P\dot{A} = \dot{Q}$, $Q\dot{A} = \dot{R}$, $R\dot{A} = \dot{S}$, &c. and the fluent of $A^n \dot{x}$ will be equal to $x A^n - n A^{n-1} P + n \times \frac{n-1}{2} A^{n-2} Q - n \times \frac{n-1}{2} \times \frac{n-2}{2} A^{n-3} R - \&c.$ For, by art. 738. the fluent of $A^n \dot{x}$ is $x A^n - F$, $n A^{n-1} \dot{x} \dot{A}$ (where F is prefixed to denote the fluent of the expression that immediately follows) $= x A^n - F$, $n A^{n-1} \dot{P} = x A^n - n A^{n-1} P + F$, $n \times \frac{n-1}{2} \times A^{n-2} \dot{Q} = x A^n - n A^{n-1} P + n \times \frac{n-1}{2} \times A^{n-2} Q - F$, $n \times \frac{n-1}{2} \times \frac{n-2}{2} \times A^{n-3} \dot{Q}$, and so on.

817. For example, let $\dot{A} = \frac{x}{\sqrt{1+xx}}$, and K the fluent of $A \dot{x}$ will be $x A - B =$ (because $\dot{B} = x \dot{A} = \frac{x \dot{x}}{\sqrt{1+xx}}$, and $B =$

P p p p

=

$= \mp \sqrt{1 \mp xx} \times A \pm \sqrt{1 \mp xx}$; and, because the fluents B, C, D, &c. are express'd by circular arks or logarithms with algebraic quantities, according as A is itself a circular ark or logarithm, the same is to be said of the fluents K, L, M, N, &c. by art. 814. Let $\sqrt{1 \mp xx} = z$; then $\dot{A} = \frac{\dot{x}}{z}$, $\dot{P} = x\dot{A} = \frac{x\dot{x}}{z} = \mp \dot{z}$, and $P = \mp z$; $\dot{Q} = P\dot{A} = \mp \dot{x}$ and $Q = \mp x$; $\dot{R} = Q\dot{A} = \mp \frac{x\dot{x}}{z} = \mp \dot{z}$, and $R = z$; $\dot{S} = R\dot{A} = \dot{x}$, and $S = x$. Therefore the fluent of $A^n \dot{x}$ is $x A^n \pm n A^{n-1} z \mp n \times \frac{n-1}{2} \times A^{n-2} x - n \times \frac{n-1}{2} \times \frac{n-2}{2} \times A^{n-3} z + n \times \frac{n-1}{2} \times \frac{n-2}{2} \times \frac{n-3}{2} \times A^{n-4} x + \&c.$

818. Supposing $\dot{A} = y\dot{x}$, $\dot{B} = x\dot{A} = yxx$. If y can be expressed by x , $\dot{B}$ may be expressed by a fluxion that involves an invariable quantity only (*viz.* x) with its fluxion; and if A and B can be reduced to algebraic quantities, or to circular arks or logarithms by the preceeding articles, the same is to be said of K the fluent of $A\dot{x}$; because $K = xA - B$. It is obvious that if A and x be assignable by each other, $A\dot{x}$ or K may be easily expressed by a fluxion that involves one variable quantity only (*viz.* x or A) with its fluxion; and the fluent of $A\dot{x}$ may in many Cases be assigned in algebraic quantities, or compared with circular arks or logarithms by the preceeding articles. But besides these more obvious cases, there are others wherein the fluent of $x \times F$, $y\dot{x}$ (or of $A\dot{x}$) can be reduced to such as have been considered above.

819. The base of a figure being represented by z and the ordinate by y , let $z = \dot{x}x^{r-1} \times \frac{E + Fx^l}{k}$ and $y = \dot{x}x^{s-1} \times \frac{e + fx^n}{k}$, and let $x = d$ when $E + Fx^n = 0$, (that is, let $d^n = -\frac{E}{F}$;) then if $r + s + l + k = 0$, the area of the figure (or the fluent of zy ;) that is generated while x by flowing from 0 becomes equal to d , shall be equal to the simultaneous

Fig. 302.

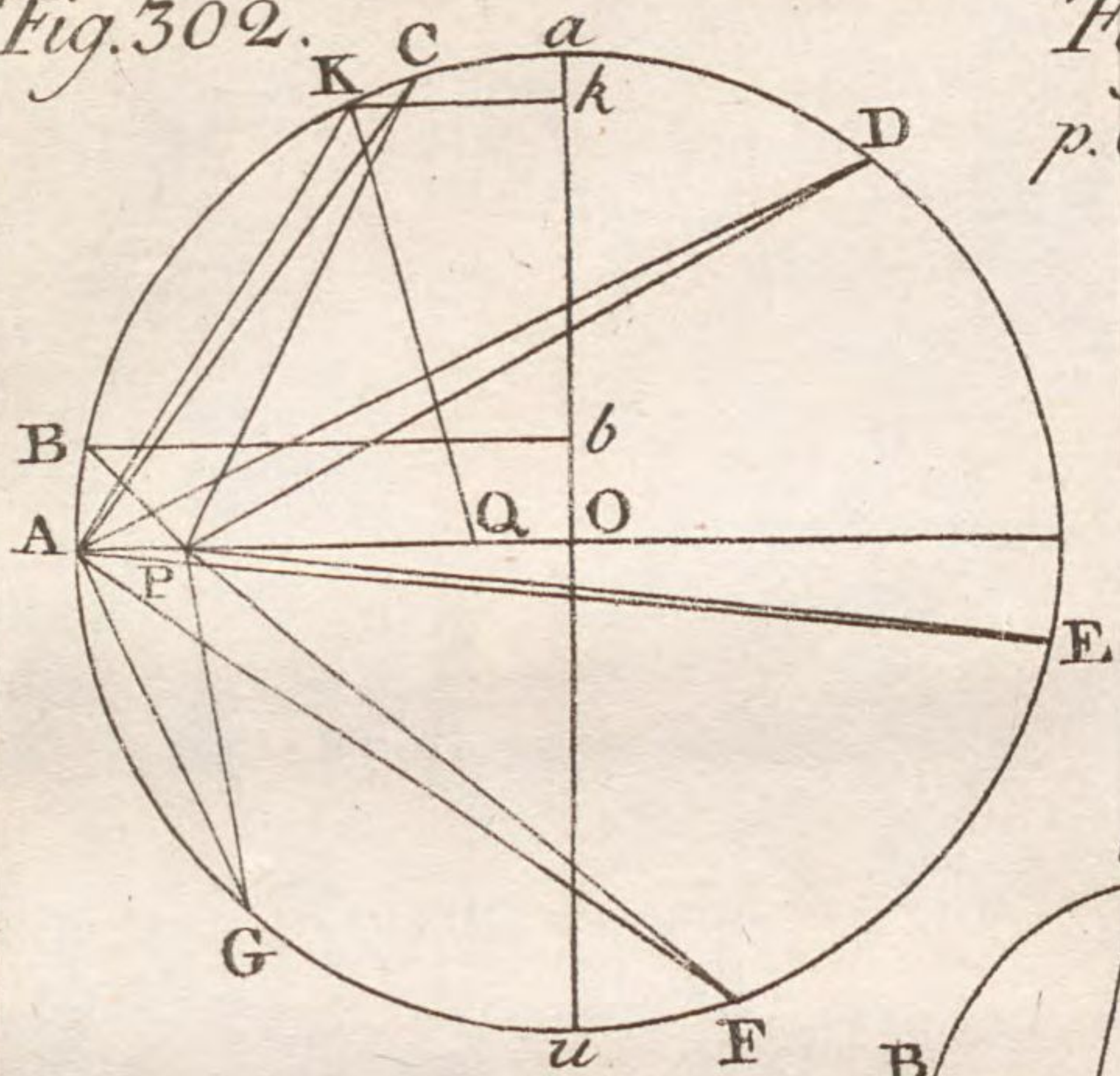


Fig. 301. N. 3.

p. 639. Q

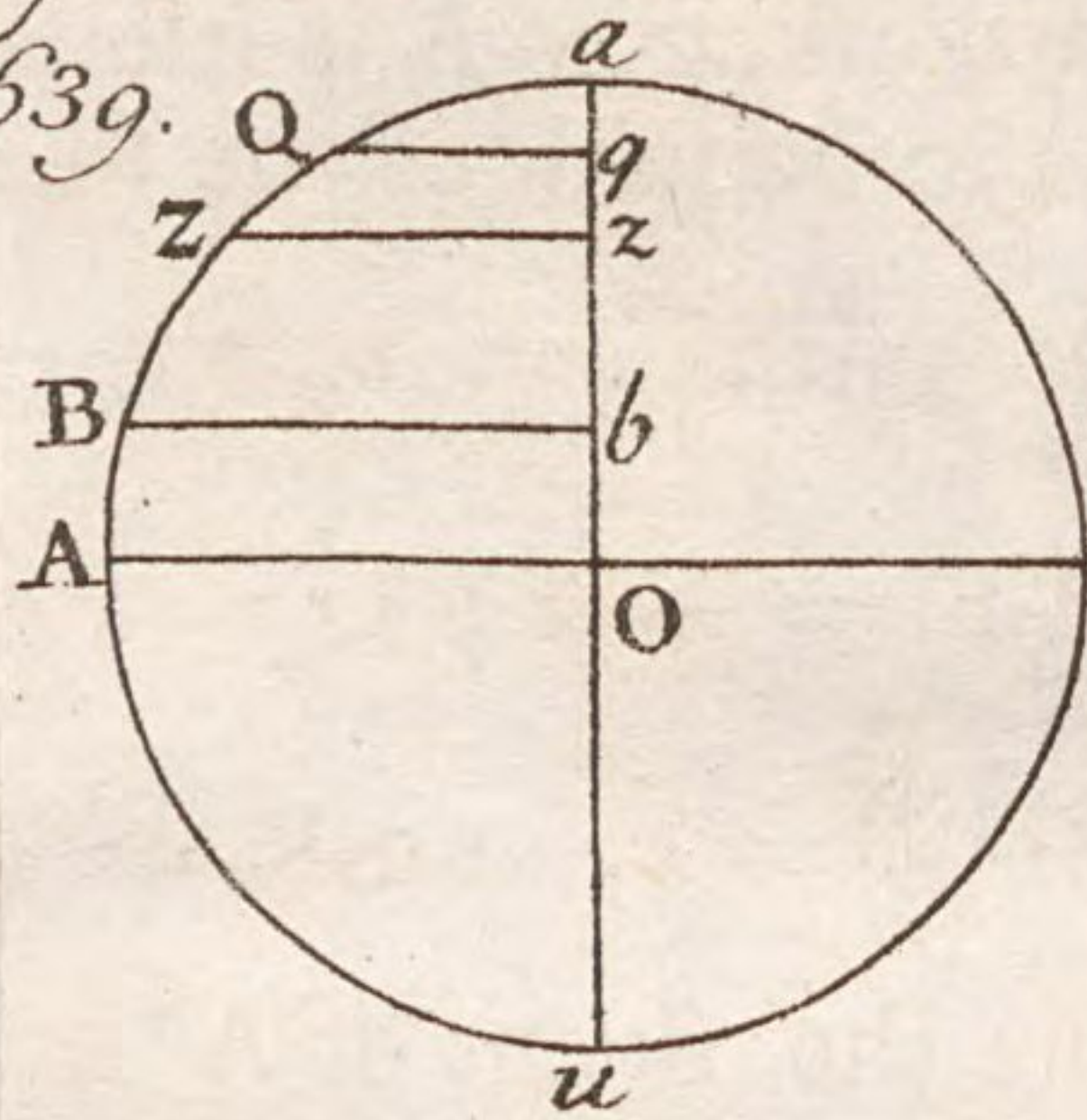


Fig. 303. n. 1. Pl. XXXV Pa. 656.

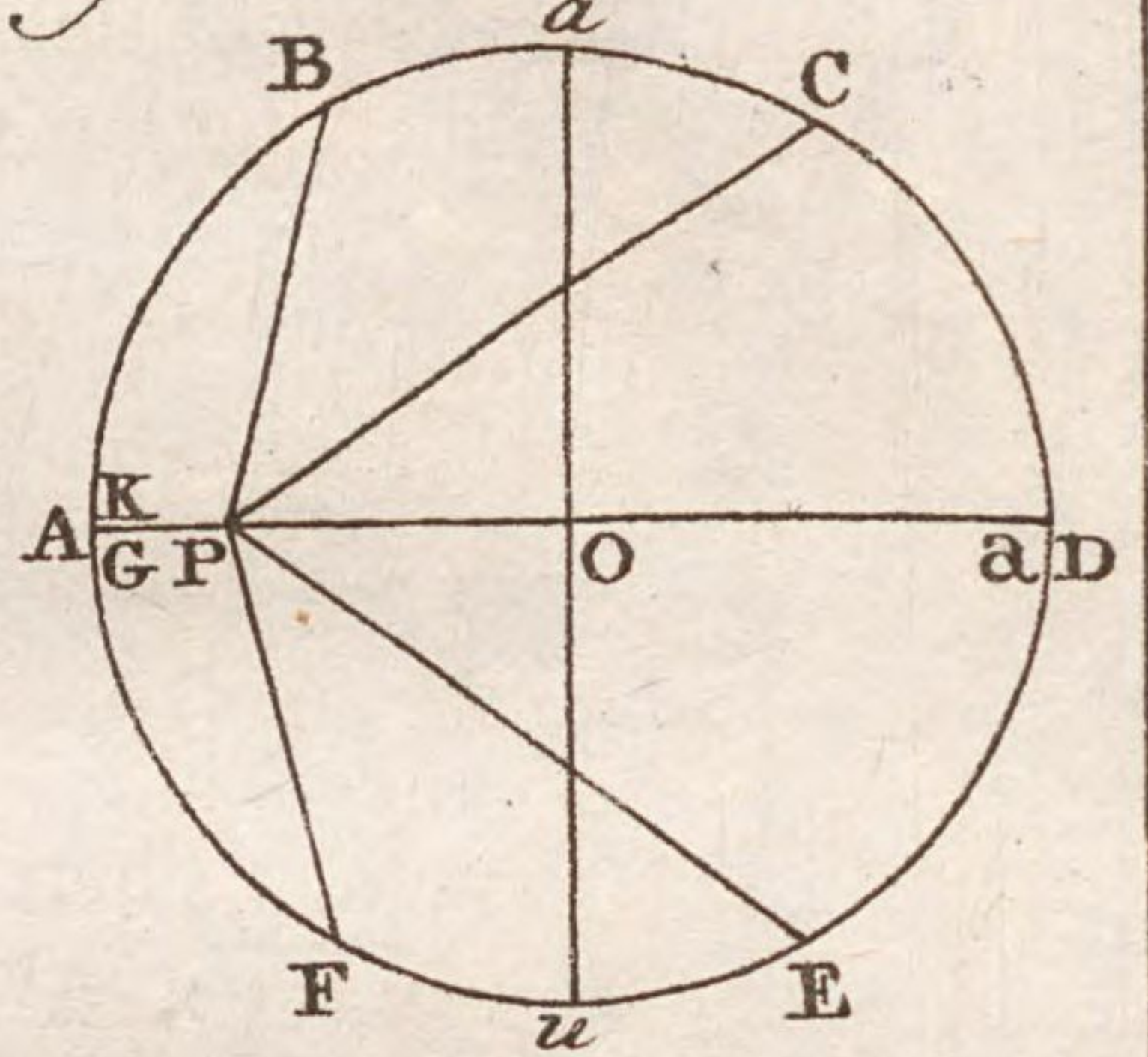


Fig. 304.

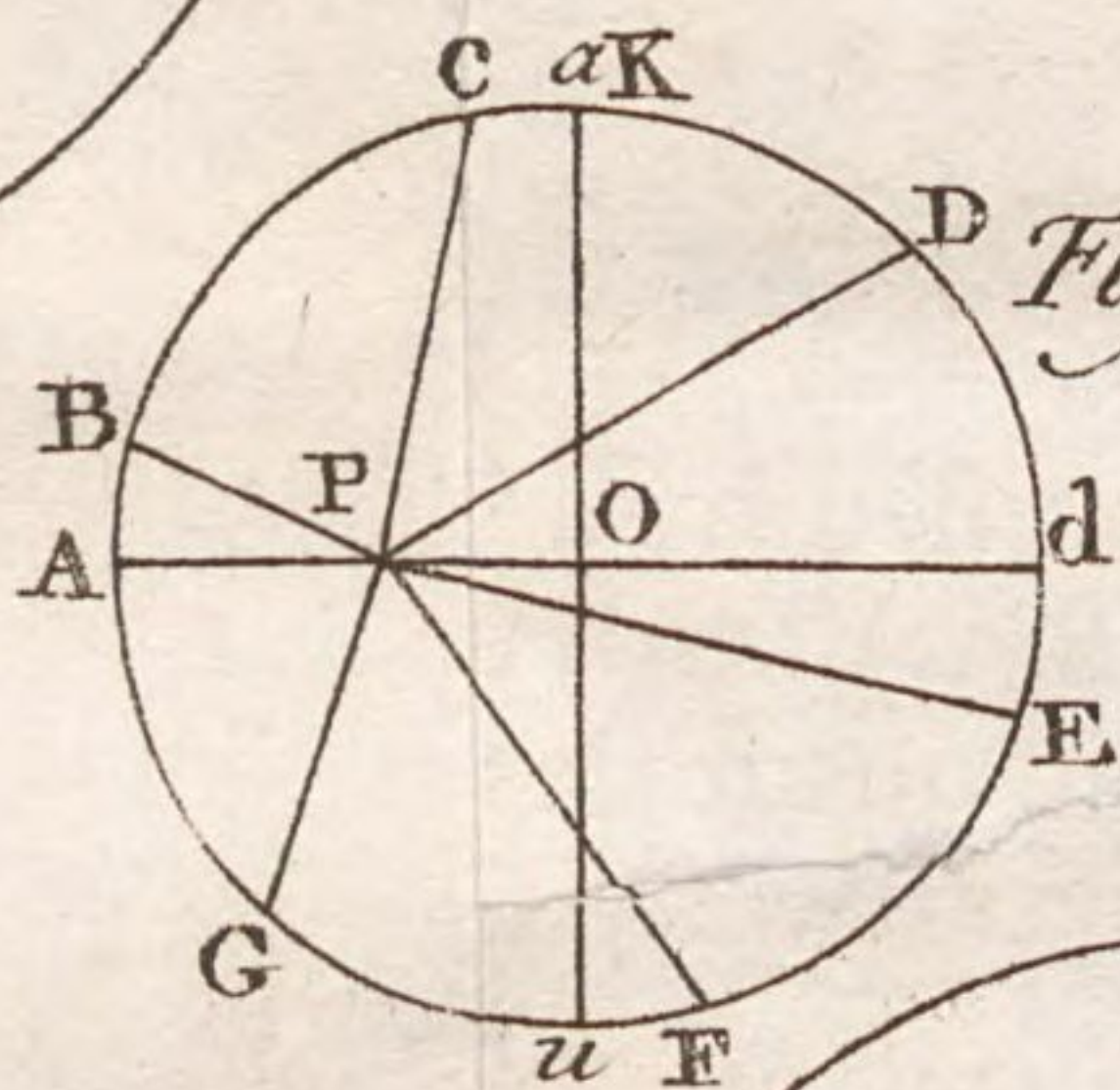


Fig. 303. N. 2.

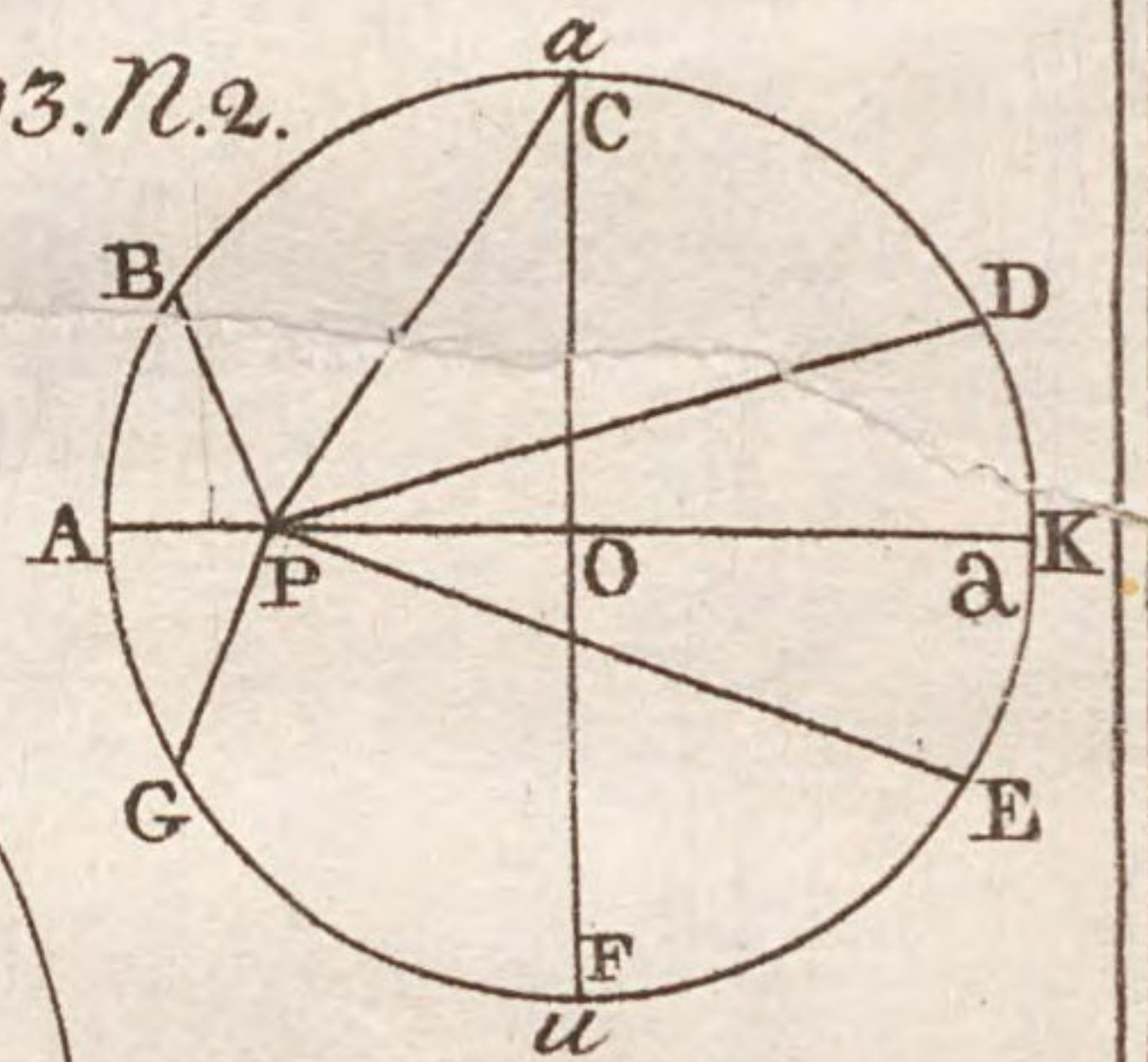


Fig. 307.

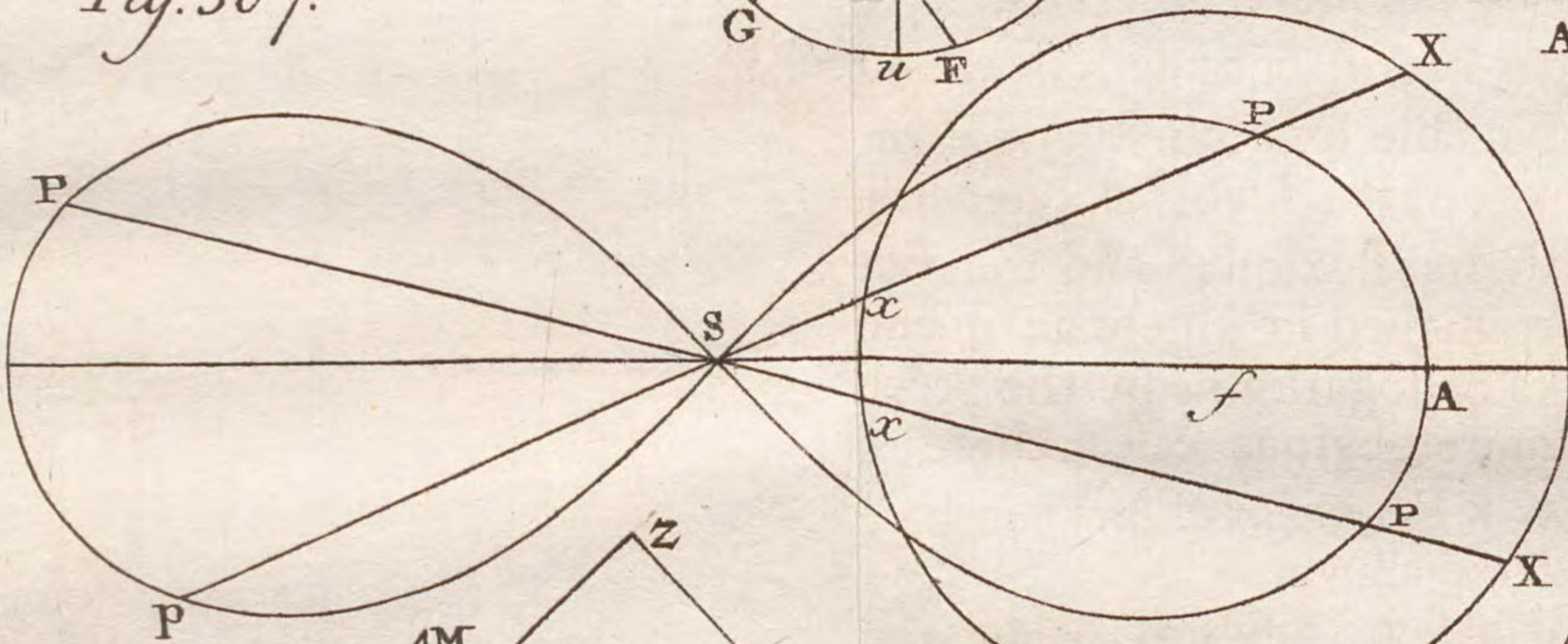


Fig. 305.
N. 1.

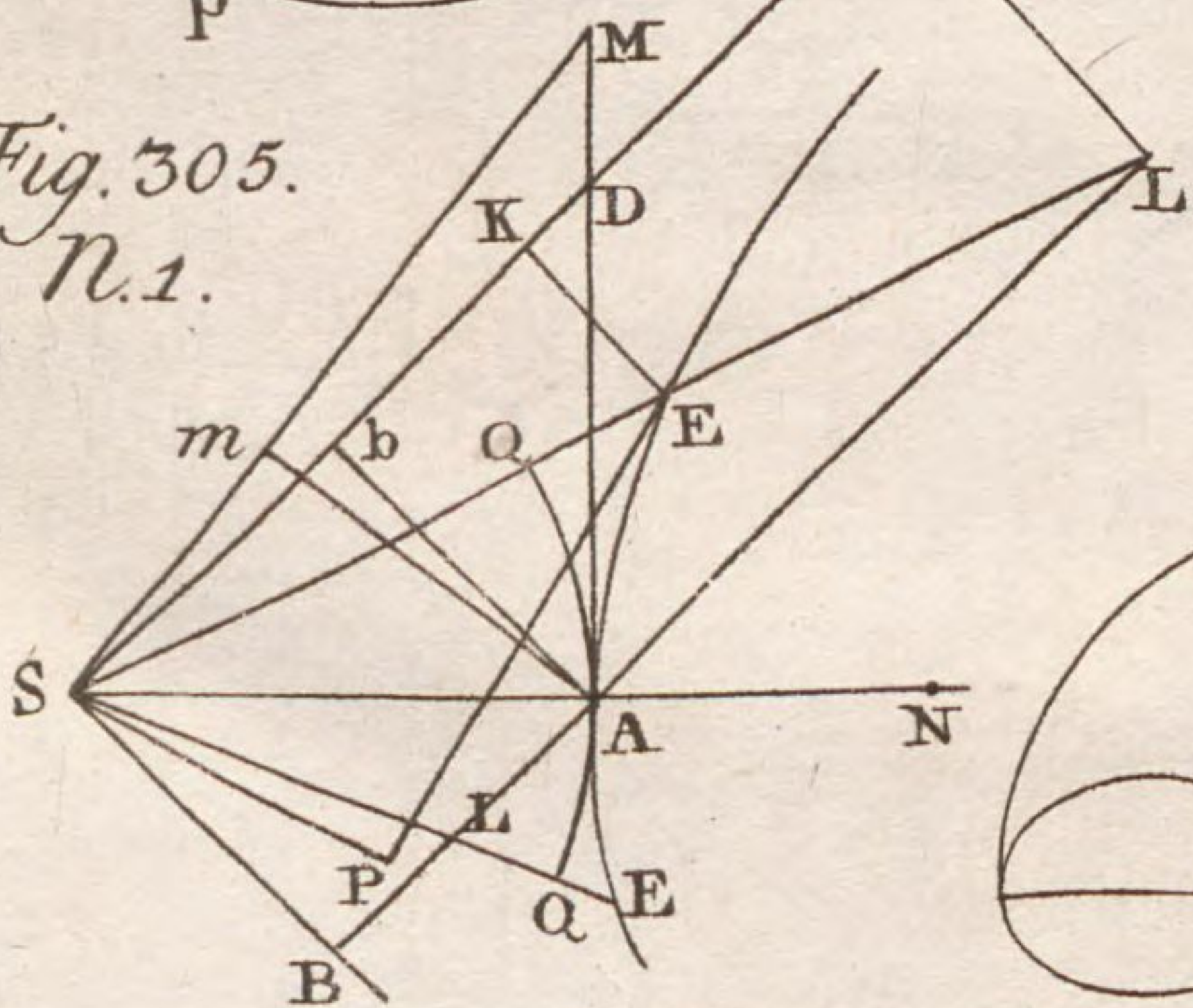


Fig. 306.

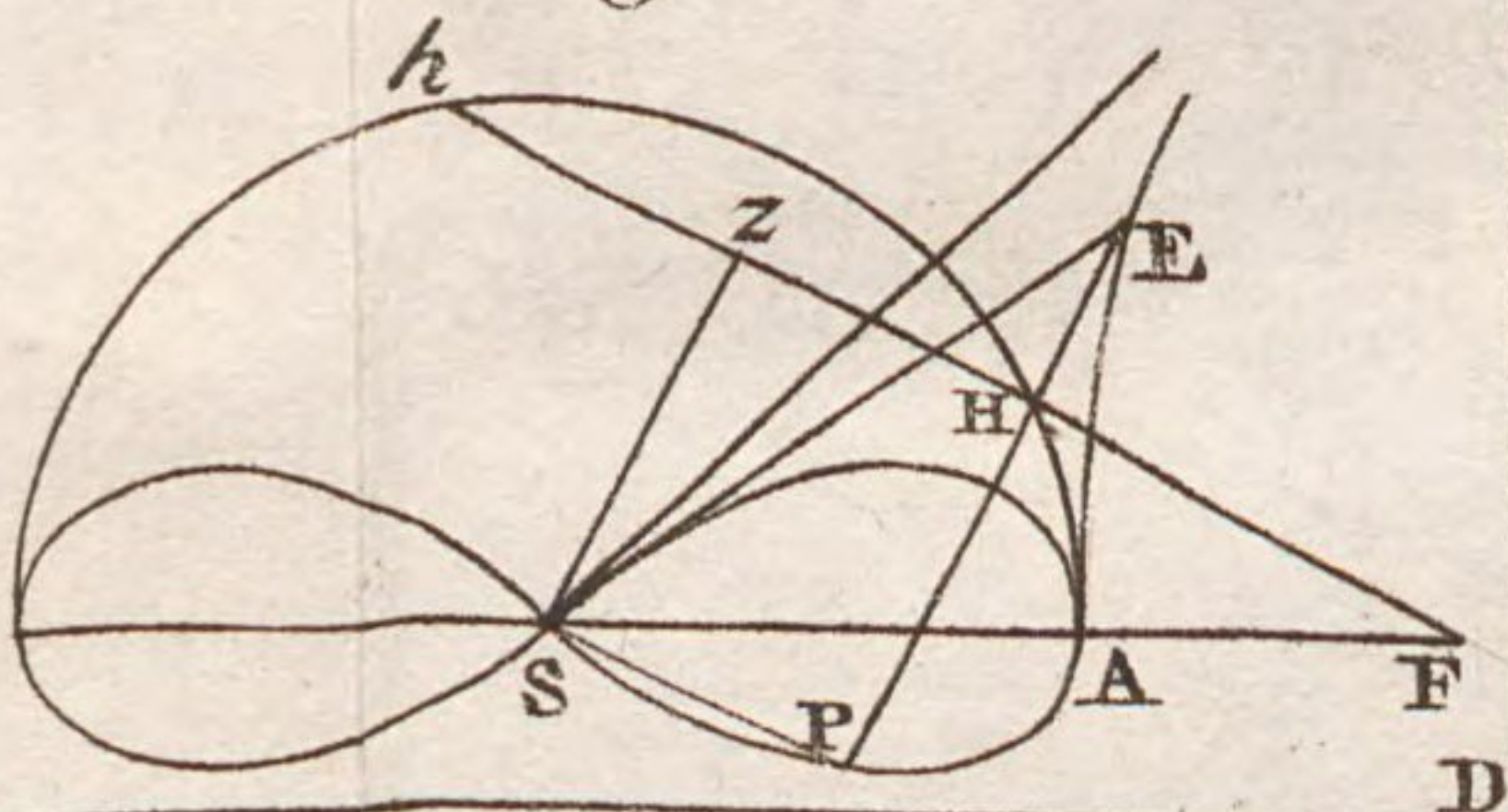
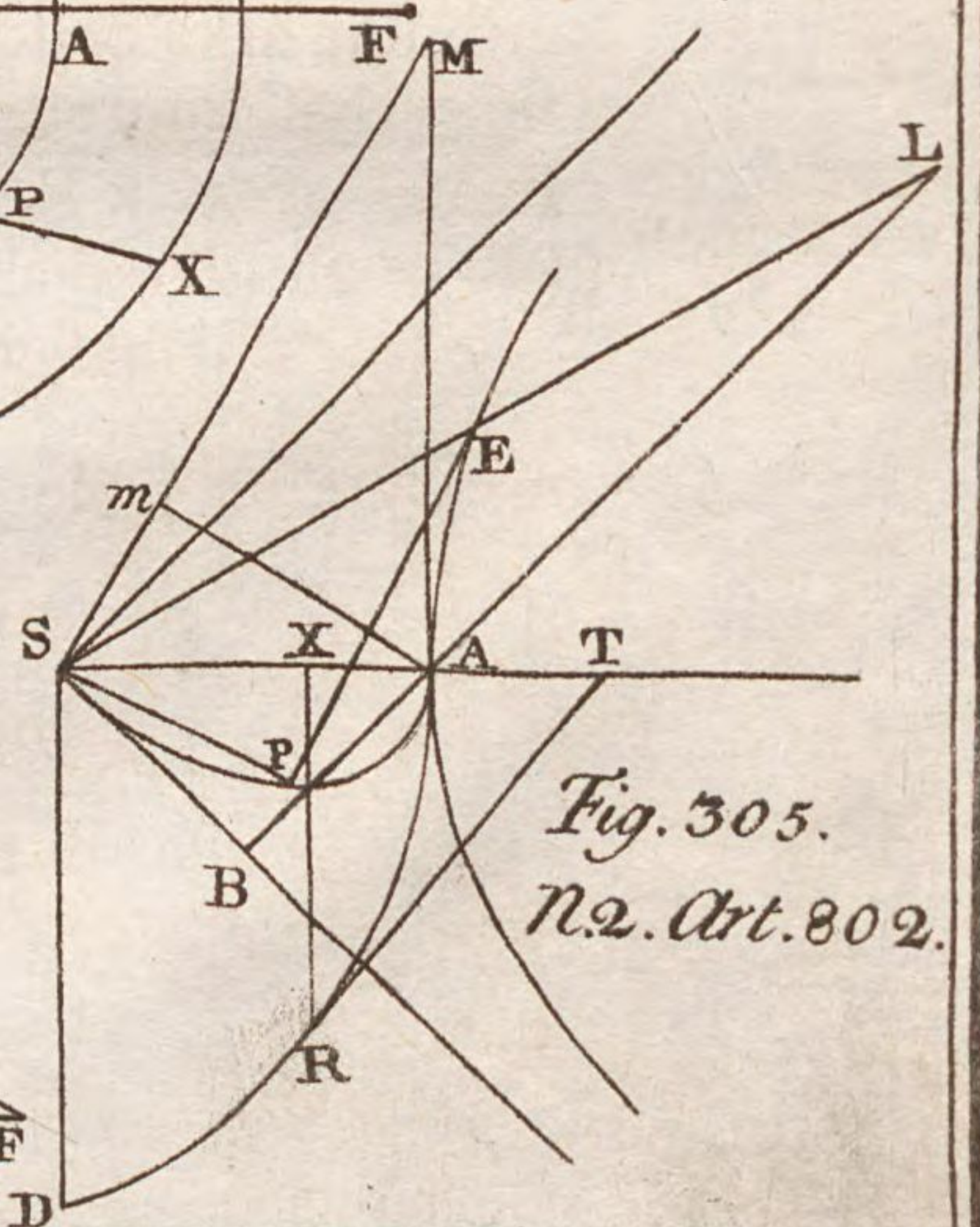


Fig. 305.
N. 2. Art. 802.



The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \frac{1}{x} \int_0^x f(t) dt$. It is shown that $f(x)$ is a continuous function and that it satisfies the differential equation $x f'(x) + f(x) = 0$. The general solution of this equation is $f(x) = \frac{C}{x}$, where C is a constant. It is then shown that $C = 1$ and that $f(x) = \frac{1}{x}$ is the unique solution of the equation.

In the second part of the paper, the function $f(x)$ is extended to the complex plane. It is shown that $f(z)$ is a meromorphic function with a simple pole at $z = 0$ and that it satisfies the functional equation $f(z) = \frac{1}{z} \int_0^z f(t) dt$. The residue of $f(z)$ at $z = 0$ is found to be 1.

The third part of the paper is devoted to the study of the asymptotic behavior of $f(x)$ as $x \rightarrow \infty$. It is shown that $f(x) \sim \frac{1}{x}$ as $x \rightarrow \infty$.

The fourth part of the paper is devoted to the study of the asymptotic behavior of $f(x)$ as $x \rightarrow 0$. It is shown that $f(x) \sim \frac{1}{x}$ as $x \rightarrow 0$.

The fifth part of the paper is devoted to the study of the asymptotic behavior of $f(x)$ as $x \rightarrow \infty$ and $x \rightarrow 0$. It is shown that $f(x) \sim \frac{1}{x}$ as $x \rightarrow \infty$ and $x \rightarrow 0$.

The sixth part of the paper is devoted to the study of the asymptotic behavior of $f(x)$ as $x \rightarrow \infty$ and $x \rightarrow 0$. It is shown that $f(x) \sim \frac{1}{x}$ as $x \rightarrow \infty$ and $x \rightarrow 0$.

The seventh part of the paper is devoted to the study of the asymptotic behavior of $f(x)$ as $x \rightarrow \infty$ and $x \rightarrow 0$. It is shown that $f(x) \sim \frac{1}{x}$ as $x \rightarrow \infty$ and $x \rightarrow 0$.

The eighth part of the paper is devoted to the study of the asymptotic behavior of $f(x)$ as $x \rightarrow \infty$ and $x \rightarrow 0$. It is shown that $f(x) \sim \frac{1}{x}$ as $x \rightarrow \infty$ and $x \rightarrow 0$.

The ninth part of the paper is devoted to the study of the asymptotic behavior of $f(x)$ as $x \rightarrow \infty$ and $x \rightarrow 0$. It is shown that $f(x) \sim \frac{1}{x}$ as $x \rightarrow \infty$ and $x \rightarrow 0$.

The tenth part of the paper is devoted to the study of the asymptotic behavior of $f(x)$ as $x \rightarrow \infty$ and $x \rightarrow 0$. It is shown that $f(x) \sim \frac{1}{x}$ as $x \rightarrow \infty$ and $x \rightarrow 0$.

ous fluents of $x^{rn+sn-1} \dot{x} \times \overline{E+F x^{n-1}}$ and $x^{sn-1} \dot{x} \times \overline{e+f x^{n-1}}$ multiplied by $\frac{1}{e^l d^{sn}}$. That is, let Q, G and P represent the fe-

veral fluents of $\dot{x} x^{rn-1} \times \overline{E+F x^{n-1}} \times F$, $\dot{x} x^{sn-1} \times \overline{e+f x^{n-1}}$, $\dot{x} x^{rn+sn-1} \times \overline{E+F x^{n-1}}$, and $\dot{x} x^{sn-1} \times \overline{e+f x^{n-1}}$ that are generated while x by flowing from o becomes equal to d ; and

$Q = \frac{GP}{e^l d^{sn}}$. For, by the supposition, $\frac{y}{e^k} = \dot{x} x^{sn-1} \times$

$\overline{1 + \frac{f x^{n-1}}{e}} =$ (by the binomial theorem) $x^{sn-1} \dot{x} + \frac{kf}{e} x$

$x^{sn+n-1} \dot{x} + k \times \frac{k-1}{2} \times \frac{ff}{ee} x^{sn+2n-1} \dot{x} + \&c.$ and (A)

$\frac{sm y}{e^k x^{sn}} = 1 + \frac{s}{s+1} \times \frac{kf x^n}{e} + \frac{s}{s+2} \times k \times \frac{k-1}{2} \times \frac{ff x^{2n}}{ee} + \&c.$

consequently $y \dot{x}$ is equal to the product of $\frac{e^k \dot{x}}{sn} \times x^{rn+sn-1}$

$\times \overline{E+F x^{n-1}}$ multiplied by this series. Therefore, by art. 792. if in this series you substitute d for x , and multiply the terms

respectively by $1, \frac{r+s}{r+s+l}, \frac{r+s}{r+s+l} \times \frac{r+s+1}{r+s+l+1}, \&c.$ or (because

$r+s = -l-k$, and $r+s+l = -k$) by $1, \frac{l+k}{k}, \frac{l+k}{k} \times$

$\frac{l+k-1}{k-1}, \&c.$ and suppose the series thence arising, viz. $1 +$

$\frac{s}{s+1} \times k + l \times \frac{f d^n}{e} + \frac{s}{s+2} \times k + l \times \frac{k+l-1}{2} \times \frac{ff d^{2n}}{ee} + \&c. =$

L , we shall have $Q = \frac{e^k}{sn} \times GL$. But by substituting in the

equation A, by which the value of y was determined, P

for y , $k+l$ for k , and d for x , it is manifest that $L = \frac{snP}{e^{k+l} d^{sn}}$;

consequently $Q = \frac{GP}{e^l d^{sn}}$. This theorem is founded on art.

792. and is to be understood with similar limitations, particularly

ly with those described in art. 796. We have supposed $r + s + l + k = 0$, or $s = -r - l - k$; but it is easy to see that if s be increased or diminished by any integer number, this theorem will be of use for discovering the fluent of yz , when $x = d$, or for reducing it to common fluents, that is, to such as involve the powers of one variable quantity compounded together in common algebraic terms with the fluxion of that quantity. Suppose, for example, that $\dot{Q} = \dot{z} \times F$, $y x^n$, and let $e + f x^n = R$, then $\dot{Q} = \frac{\dot{z} x^{sn} R^{k+1}}{n f x^{s+k+1}} - \frac{s e y \dot{z}}{f x^{s+k+1}}$.

820. Let $x = D$ when $e + f x^n = 0$; and the values of $\dot{z}$ and $\dot{y}$ remaining the same as in the last article, let Y , Z and q be the respective fluents of $\dot{y}$, $\dot{z}$ and $y \dot{z}$ when $e + f x^n = 0$. Let g and p be the simultaneous fluents of $\dot{x} x^{rn+sn-1} \times \frac{1}{e + f x^n^k}$ and $\dot{x} x^{rn-1} \times \frac{1}{E + F x^n^{l+k}}$. Then if $r + s + l + k = 0$ as formerly, $q = YZ - \frac{gp}{E^{k+1} D^{rn}}$. This theorem follows

from the last, because $F, y \dot{z} = y \dot{z} - F, z \dot{y}$.

821. Let $\dot{z} = -\dot{x} x^{n-m-2} \times \frac{1}{E x^n + F^{l-1}}$, $\dot{y} = -\dot{x} x^m \times \frac{1}{e x^n + f^k}$, $x = d$ when $E + F x^n = 0$, and the area or fluent of $y \dot{z}$ which is generated while x flows till from being equal to d it becomes infinitely great (or the limit to which this area approximates while x increases continually) is the product of the fluents of $-\dot{x} x^{n+nk-1} \times \frac{1}{E x^n + F^{l-1}}$ and $-\dot{x} x^{m-nl} \times \frac{1}{e x^n + f^k}$ multiplied by each other and by $\frac{1}{e^l d^{m+nk+1}}$; as will appear by substituting x^{-1} for x in art. 819. and supposing $m = rn + ln - 1$. The fluent of $y \dot{z}$ when $e + f x^n = 0$ is the excess of the product of the corresponding values of y and z above the product of the simultaneous fluents of $-\dot{x} x^{nl-1} \times \frac{1}{e x^n + f^k}$ and $-\dot{x} x^{-m-nk-2} \times \frac{1}{E x^n + F^{l+k}}$ multiplied by each other and by $\frac{D^{m+1-nl}}{E^{k+1}}$, by the last article.

822. From these theorems Tables might be computed of fluents of this kind that may be reduced to circular arks and logarithms. But we shall only give a few examples of their use. Suppose that it is required to find this area or fluent of yz , when, m being any positive number, $z = \frac{x}{x^m \sqrt{bb-xx}}$ and $y = \frac{x^{m-1} \dot{x}}{aa+xx}$; that is, let it be required to find the fluent of $\frac{x}{x^m \sqrt{bb-xx}} \times F, \frac{x^{m-1} \dot{x}}{aa+xx}$ when $x=b$. In this case, by comparing the exponents with those in art. 819. $n=2$, $r = \frac{1-m}{2}$, $l = \frac{1}{2}$, $s = \frac{m}{2}$, and $k=-1$, so that $r+l+s+k = \frac{1-m+1+m}{2} - 1 = 0$ as the theorem requires. Because in this case $\dot{G} = \frac{x}{\sqrt{bb-xx}}$, and $\dot{P} = \frac{x^{m-1} \dot{x}}{\sqrt{aa+xx}}$; it follows that if N denote the ratio of the circumference of a circle to its diameter, the fluent required is $\frac{N}{2abm} \times F. \frac{x^{m-1} \dot{x}}{\sqrt{aa+xx}}$, if b be substituted for x in the value of this last fluent after it is determined. Thus if $z = \log. \frac{bb-b\sqrt{bb-xx}}{x}$ and $y = \log. a \times \sqrt{\frac{a+x}{a-x}}$, and H represent the ark described with the radius a that has its sine equal to b , then the area required will be equal to $\frac{Nb}{2} \times H$; whence the proposition that was advanced in art. 812. follows: For in this example $z = \frac{bbx}{x\sqrt{bb-xx}}$, $y = \frac{aax}{aa-xx}$, $m=1$ and $P = F, \frac{x}{\sqrt{aa-xx}}$. If the same value of z remaining, we suppose y to be always equal to the ark described with the radius a that has its tangent equal to x , or $y = \frac{aax}{aa+xx}$, then $\dot{P} = \frac{x}{\sqrt{aa+xx}}$; and in this example

the

the fluent required is $\frac{Nba}{2} \times \log. \frac{\sqrt{aa+bb}+b}{a}$; so that the fluent, which in the former case was the product of two circular arks, is now the product of a circular ark by a logarithm. If m be any integer number, the fluent required may be measured by the areas of conic sections, and if m be equal to any of the fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, &c. it may be measured by their arks.

823. It is obvious that in other cases the fluents may be computed from the theorem besides those wherein $r + s + l + k = 0$. Suppose, for example, $z = \frac{-\dot{x}}{x^3 \sqrt{bb-xx}}$ and $y = \frac{ex}{e+fx}$.

Because $y = \dot{x} - \frac{fx^2 \dot{x}}{e+fx}$, $zy = \frac{\dot{x}}{xx \sqrt{bb-xx}} - \frac{\dot{x}}{x^3 \sqrt{bb-xx}}$

$\times F, \frac{-fx^2 \dot{x}}{e+fx}$. The fluent of the former part is assignable in algebraic terms, and the fluent of the latter (by the theorem in art. 819.) by circular arks and right lines.

824. In like manner it follows from art. 822. that if $z = \frac{-\dot{x}}{x^h \sqrt{xx-bb}}$ and $y = \frac{-x^h \dot{x}}{xx+aa}$, then the area, or fluent of yz that is generated while x flows till from being equal to b it become infinite, is $\frac{N}{2bh} \times F, \frac{x^{h-1} \dot{x}}{\sqrt{xx+aa}}$.

825. The theorems in art. 819, &c. are chiefly of use for reducing fluents to circular arks or logarithms, or to others of a more simple form, (and consequently for rendering our solutions of problems more simple and elegant than when we have immediately recourse to an infinite series,) when neither y nor z can be expressed by x in algebraic terms. But they may be of some use likewise for finding the fluent of yz when y is assigned by x .

Thus to find the fluent of $\frac{aax}{aa+xx \times \sqrt{bb-xx}}$

when $x=b$, suppose $z = \frac{\dot{x}}{\sqrt{bb-xx}}$, $y = \frac{aa}{aa+xx}$, and consequently

quently $\dot{y} = \frac{-2aax\dot{x}}{aa+xx}$. By comparing these values of $\dot{z}$ and $\dot{y}$ with their general values in art. 819. $n=2$, $r=\frac{1}{2}$, $l=\frac{1}{2}$, $s=1$, $k=-2$, and $r+l+s+k=0$, as the theorem requires, G the fluent of $\frac{x^2\dot{x}}{\sqrt{bb-xx}}$ is $\frac{Nbb}{4}$, and P the fluent of $\frac{-x\dot{x}}{aa+xx}$ is $\frac{I}{\sqrt{aa+xx}}$; whence by the theorem in art. 819. the fluent required is $\frac{Na}{2\sqrt{aa+bb}}$. Other examples might be given, if we were not obliged to hasten towards a conclusion, this treatise having already grown to a far greater bulk than was at first intended.

826. If we assume an equation as $\overline{x+Ay^m} \times \overline{x+By^n} = E$, where m, n, A, B, E are supposed invariable, then (by art. 728.) $\frac{mx + mA\dot{y}}{x+Ay} + \frac{nx + nB\dot{y}}{x+By} = 0$, and $\overline{m+n} \times x\dot{x} + \overline{nA+mB} \times y\dot{y} + \overline{m+n} \times AB\dot{y}y = 0$. If we had assumed $\overline{x+Ay^m} \times \overline{y^n} = E$, then $m\dot{y}x + nxy + \overline{m+n} \times A\dot{y}y = 0$, where the term $x\dot{x}$ is wanting. When a fluxional equation is proposed that can be reduced to a form of this kind, then, by comparing its coefficients with those of the equation of the same form, the values of $m, n, A, \&c.$ may be determined, and the equation for the fluents discovered; as is shewn more fully *comment. petropol. tom. I. &c.*

827. When an area or fluent is reduced to a series by the methods described in art. 745, &c. the series in some cases converges at so slow a rate as to be of little use for finding the area. Suppose FMf to be an equilateral hyperbola that has its center in O and Oa for one of its asymptotes; let OA = 1, AP = x, PM = y, and $y = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \&c.$ whence the area APMF = F, $y\dot{x} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.$ and if

FIG. 313.

if $AB = 1$, the area $ABEF = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \&c. = \frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \&c.$ which is the series mentioned for the quadrature of the hyperbola (or for finding the hyperbolic logarithm of 2) in art. 361. But this series converges so slowly, that the sum of the first 1000 terms* of it is found deficient from the true value of the area in the fifth decimal; and other examples similar to this might be brought, wherein the area may be more easily computed from the inscribed polygons than from the series. Some further artifice is therefore necessary in order to compute the area in such cases, instances of which were described in art. 361. and others are to be met with in several authors, particularly those who have treated of the computation of logarithms and mensuration of the circle. The following theorems, derived from the method of fluxions, may be of use for this purpose; and serve for the resolution of many problems that are usually referred to what is called Sir ISAAC NEWTON's *differential method*.

FIG. 314. 828. Suppose the base $AP = z$, the ordinate $PM = y$, and the base being supposed to flow uniformly let $\dot{z} = 1$. Let the first ordinate AF be represented by a , $AB = 1$, and the area $ABEF = A$. As A is the area generated by the ordinate y , so let $B, C, D, E, F, \&c.$ represent the areas upon the same base AB generated by the respective ordinates $\dot{y}, \ddot{y}, \ddot{\ddot{y}}, \ddot{\ddot{\ddot{y}}}, \&c.$ Then $AF = a = A - \frac{B}{2} + \frac{C}{12} - \frac{E}{720} + \frac{G}{30240} - \&c.$ For by art. 752. $A = a + \frac{\dot{a}}{2} + \frac{\ddot{a}}{6} + \frac{\ddot{\ddot{a}}}{24} + \frac{\ddot{\ddot{\ddot{a}}}}{120} + \&c.$ whence we have the equation (Q) $a = A - \frac{\dot{a}}{2} - \frac{\ddot{a}}{6} - \frac{\ddot{\ddot{a}}}{24} - \frac{\ddot{\ddot{\ddot{a}}}}{120} - \&c.$ In like manner $\dot{a} = B - \frac{\ddot{a}}{2} - \frac{\ddot{\ddot{a}}}{6} - \frac{\ddot{\ddot{\ddot{a}}}}{24} - \&c.$ $\ddot{a} = C - \frac{\ddot{\ddot{a}}}{2} - \frac{\ddot{\ddot{\ddot{a}}}}{2} - \&c.$ $\ddot{\ddot{a}} = D - \frac{\ddot{\ddot{\ddot{a}}}}{2} - \&c.$ $\ddot{\ddot{\ddot{a}}} = E - \&c.$ by which latter

* Stirling *de summat. serierum*, p. 28.

ter equations if we exterminate $\dot{a}, \ddot{a}, \dddot{a}, \&c.$ from the value of a in the equation Q , we shall find that $a = A - \frac{B}{2} + \frac{C}{12} - \frac{E}{720} + \&c.$ The coefficients are continued thus: Let $k, l, m, n, \&c.$ denote the respective coefficients of $\dot{a}, \ddot{a}, \dddot{a}, \&c.$ in the equation Q ; that is, let $k = \frac{1}{2}, l = \frac{1}{6}, m = \frac{1}{24}, n = \frac{1}{120}, \&c.$ suppose $K = k = \frac{1}{2}, L = kK - l = \frac{1}{12}, M = kL - lK + m = 0, N = kM - lL + mK - n = -\frac{1}{720},$ and so on; then $a = A - KB + LC - MD + NE - \&c.$ where the coefficients of the alternate areas $D, F, H, \&c.$ vanish.

829. As A is the fluent of $y\dot{z}$, so B is the fluent of $\dot{y}\dot{z}$, C of $\ddot{y}\dot{z}$, E of $\ddot{y}\ddot{z}$, $\&c.$ Therefore, since $\dot{z} = 1$, and these areas are generated while the ordinate PM moves from AF to BE , the area B will be express'd by the excess of the last ordinate BE above the first AF , C by the difference of the fluxions of the ordinates (having due regard to the signs of these fluxions,) E by the difference of their third fluxions, and the other areas $G, I, \&c.$ by the respective differences of their fluxions of the corresponding higher orders. Therefore if α represent $BE - AF$ the difference of the ordinates, and $\beta, \delta, \zeta, \&c.$ the differences of their fluxions of the first, third, fifth orders, $\&c.$ then $a = A - \frac{\alpha}{2} + \frac{\beta}{12} - \frac{\delta}{720} + \frac{\zeta}{30240} + \&c.$

830. Supposing now the base Aa to be divided into the equal successive parts $AB, BC, CK, \&c.$ and each part equal to unit, let the sum of the equidistant ordinates $AF, BE, CK, \&c.$ exclusive of the last ordinate af be represented by S , the total area $AFfa$ upon the base Aa by A , the excess of af above AF by a , the respective excesses of their first, third, fifth fluxions, $\&c.$ by $\beta, \delta, \zeta, \&c.$ the fluxion of the base being supposed equal to 1, then it follows from the last article that $S = A - \frac{a}{2} + \frac{\beta}{12}$

Q q q q

$-\frac{\delta}{720} + \frac{\zeta}{30240} + \&c.$ and $A = S + \frac{a}{2} - \frac{\beta}{12} + \frac{\delta}{720} - \frac{\zeta}{30240} + \&c.$ which give two of the theorems mentioned in art. 352. and 353. where we had hyperbolic figures chiefly in view. This proposition more generally express'd, without supposing z or z equal to unit, is that $S = \frac{A}{z} - \frac{a}{2} + \frac{z^3}{12z} - \frac{z^5}{720z^3} + \frac{z^7}{30240z^5} - \frac{z^9}{1209600z^7} + \&c.$

831. The ordinate AF being still represented by a , let AR and Ar be taken on opposite sides of the point A equal to each other, RV and rv the ordinates at R and r terminate the area RVvr; let y represent any ordinate as PM of the figure, and the base being supposed to flow uniformly let A, C, E, &c. represent the areas upon the base Rr that are generated by the respective ordinates $y, \ddot{y}, \ddot{y}, \&c.$ Then supposing AR = z , the middle ordinate AF ($=a$) = $\frac{A}{2z} - \frac{zC}{12z^2} + \frac{7z^3E}{720z^4} - \frac{31z^5G}{30240z^5} + \&c.$ For by art. 752. $\frac{RrvV}{2z} = \frac{A}{2z} = a + \frac{z^2\ddot{a}}{2 \times 3z^2} + \frac{z^4\ddot{\ddot{a}}}{2 \times 3 \times 4 \times 5z^4} + \&c.$ or $a = \frac{A}{2z} - \frac{z^2\ddot{a}}{6z^2} - \frac{z^4\ddot{\ddot{a}}}{120z^4} - \&c.$ In like manner $\ddot{a} = \frac{C}{2z} - \frac{z^2\ddot{\ddot{a}}}{6z^2} - \&c.$ $\ddot{\ddot{a}} = \frac{E}{2z} - \&c.$

whence by exterminating $\ddot{a}, \ddot{\ddot{a}}, \&c.$ from the value of a , the theorem will appear. The coefficients of C, E, G, &c. are continued thus; let the several coefficients of $\ddot{a}, \ddot{\ddot{a}}, \&c.$ in the value of $\frac{A}{2z}$ (derived from art. 752.) be represented by $k, l, m, n, \&c.$ that is, let $k = \frac{z^2}{2 \times 3z^2}, l = \frac{z^4}{4 \times 5z^2}, m = \frac{z^6}{6 \times 7z^2}, n = \frac{z^8}{8 \times 9z^2}, \&c.$ then let $K = \frac{k}{2z} = \frac{z}{12z^2}, L = kK -$

$$\frac{l}{2z},$$

$$\frac{l}{2z}, M = kL - lK + \frac{m}{2z}, N = kM - lL + mK - \frac{n}{2z}, \&c.$$

And the values of the coefficients K, L, M, N, &c. being thus computed, then $a = \frac{A}{2z} - KC + LE - MG + NI - \&c.$ Be-

cause the areas C, E, &c. are the respective fluents of $\ddot{y}z$, $\ddot{y}z$, &c. if the respective differences of the first, third, fifth, seventh and higher alternate fluxions of the ordinates rv and RV be expressed by $\beta, \delta, \zeta, \theta, \&c.$ then $a = \frac{A}{2z} - \frac{z\beta}{12z} +$

$$\frac{7z^3\delta}{720z^3} - \frac{31z^5\zeta}{30240z^5} + \frac{127z^7\theta}{1209600z^7} - \&c.$$

832. From this it follows, that if AF, BE, CK, &c. be a series of equidistant ordinates upon the base Aa , of which AF is the first and af the last; AB their common distance be equal to $2z$; AR be taken backwards from A equal to z or $\frac{1}{2}$ AB, and ar be taken forwards from a also equal to $\frac{1}{2}$ AB; the ordinates RV and rv terminate the area $RVvr$; and this area being represented by A, the differences by which the first, third, fifth, seventh and higher alternate fluxions of rv exceed the same fluxions of RV be express'd by $\beta, \delta, \zeta, \theta, \&c.$ and the sum of the ordinates AF, BE, CK, &c. (including af) by S, then

$$S = \frac{A}{2z} - \frac{z\beta}{12z} + \frac{7z^3\delta}{720z^3} - \frac{31z^5\zeta}{30240z^5} + \frac{127z^7\theta}{1209600z^7} - \frac{511z^9\kappa}{47900160z^9} + \&c. \quad A = 2zS + \frac{z^2\beta}{6z} + \frac{7z^4\delta}{360z^3} - \frac{31z^6\zeta}{15120z^5} + \frac{127z^8\theta}{604800z^7} - \frac{511z^{10}\kappa}{23950080z^9} + \&c.$$

If we suppose $AB = 1$, and $z = 1$, then $z = \frac{1}{2}$ and $S = A - \frac{\beta}{24} + \frac{7\delta}{5760} - \frac{31\zeta}{967680} + \frac{127\theta}{154828800} - \&c.$ and $A = S + \frac{\beta}{24} - \frac{7\delta}{5760} + \frac{31\theta}{967680} - \&c.$ which are the two other theorems mentioned in art. 352. and 353. only, in order to include the term af , ar is here taken forwards from a , whereas af was there excluded and ar taken the contrary way.

Q q q q 2

833. The

833. The use of these theorems will best appear by examples. First, let $m, m + e, m + 2e, m + 3e, \dots n$, be a series of numbers in arithmetical progression, where m denotes the first term, e the common difference, and n the last term; and r being any number positive or negative (-1 excepted) S the sum of the powers of these numbers of the exponent r , that is, $m^r + m^r + e^r + m^r + 2e^r + m^r + 3e^r + \dots + n^r = \frac{1}{r+1} \times n^{r+1} - m^{r+1}$

$$+ \frac{n^r + m^r}{2} + \frac{re}{12} \times n^{r-1} - m^{r-1} - \frac{r \cdot r-1 \cdot r-2 \cdot e^3}{720} \times$$

FIG. 314.
N. 1. & 2.

$n^{r-3} - m^{r-3} + \&c.$ For supposing $OP = x$, $PM = y$, let FMf be the parabola or hyperbola whose equation is $y = x^r$, $OA = m$, $Oa = n$; consequently $Af = m^r$, $af = n^r$, $F.yx = F.x^m \dot{x} = \frac{x^{m+1}}{m+1}$ and the area $AFfa = A = \frac{n^{r+1} - m^{r+1}}{r+1}$; $af - AF = a = n^r - m^r$; $\dot{y} = rx^{r-1}\dot{x}$, and supposing $\dot{x} = \dot{z} = 1$, the difference of the fluxions of af and AF is $rn^{r-1} - rm^{r-1} = \beta$; $\dot{y} = r \times r-1 \times r-2 \times x^{r-3}\dot{x}$, and $\delta = r \times r-1 \times r-2 \times n^{r-3} - m^{r-3}$. In like manner $\zeta, \theta, \&c.$ are computed, and it follows from art. 830. that $S - n^r = \frac{n^{r+1} - m^{r+1}}{r+1} \times e$

$$- \frac{n^r - m^r}{2} + \frac{re}{12} \times n^{r-1} - m^{r-1} - \&c. \text{ therefore } S = \frac{n^{r+1} - m^{r+1}}{r+1} \times e + \frac{n^r + m^r}{2} + \frac{re}{12} \times n^{r-1} - m^{r-1} - \&c. \text{ By}$$

supposing $e = 1$ and $m = 0$, it follows that the sum of the powers of the numbers $0, 1, 2, 3, 4, \dots n$ of any integer and positive exponent r is $\frac{n^{r+1}}{r+1} + \frac{n^r}{2} + \frac{rn^{r-1}}{12} - \frac{r \times r-1 \times r-2 \times n^{r-3}}{720} + \&c.$ this series being continued to as many terms as there are units in $2 + \frac{r-1}{2}$ only, when r is an odd number: because when $r = 1$, the fluxions of AF and af are equal, and $\beta = 0$

$\beta = 0$; when $r = 3$, $\delta = 0$; when $r = 5$, $\zeta = 0$, &c. Thus if $r = 1$, $S = \frac{n^2}{2} + \frac{n}{2}$; if $r = 2$, $S = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$, and if $r = 3$, $S = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$. This is the theorem given by Mr. JAMES BERNOULLI. *Ars conjectandi*, p. 97. When r is a fraction or negative number, the sum of the powers of the same numbers (by supposing $m = 1$) is $\frac{n^{r+1}-1}{r+1} + \frac{n^r+1}{2} + \frac{rn^{r-1}-r}{12} - \frac{r \times r-1 \times r-2}{720} \times n^{r-3} - 1 + \&c.$

834. The sum S of the same powers of $m+e$, $m+3e$, $m+5e$, $m+7e$, $\dots$, $n-e$, where $2e$ is the common difference of the terms, computed by the theorem in art. 832. by supposing $OR = m$, $RA = e = ar$, $Or = n$, (and computing the area $RVvr$ with the differences of the first, third, and higher alternate fluxions of rv and RV) is $\frac{1}{r+1 \times 2e} \times n^{r+1} - m^{r+1} - \frac{re}{12} \times$

$$\frac{n^{r-1} - m^{r-1}}{720} + \frac{7r \times r-1 \times r-2 \times e^3}{720} \times n^{r-3} - m^{r-3} - \frac{31r \times r-1 \times r-2 \times r-3 \times r-4}{30240} \times e^5 \times n^{r-5} - m^{r-5} + \&c.$$

By supposing $m = e = \frac{1}{2}$, the numbers are 1, 2, 3, 4, 5, $\dots$

$$n - \frac{1}{2}, \text{ and } S = \frac{n^{r+1}}{r+1} - \frac{rn^{r-1}}{24} + \frac{7r \times r-1 \times r-2 \times n^{r-3}}{5760} - \frac{31r \times r-1 \times r-2 \times r-3 \times r-4}{967680} \times n^{r-5} + \&c. - \frac{1}{r+1 \times 2r+1} + \frac{r}{24 \times 2r-1} - \frac{7r \times r-1 \times r-2}{5760 \times 2r-3} + \&c.$$

835. When r is negative, let $r = -s$; and if s be greater than unit, then, by what we have shewn in article

$$833. \text{ the sum of the progression } \frac{1}{m^s} + \frac{1}{m+e^s} + \frac{1}{m+2e^s} + \frac{1}{m+3e^s} + \&c. \text{ (by supposing } \frac{1}{n^s} = 0) = \frac{1}{s-1 \times em^{s-1}} + \frac{1}{2m^s} +$$

$$+ \frac{se}{12ms+1} - \frac{s \times s + 1 \times s + 2 \times e^3}{720ms+3} + \frac{s \times s + 1 \times s + 2 \times s + 3 \times s + 4}{30240ms+5}$$

$\times e^5 - \&c.$ This series was deduced from different principles by Mr. DE MOIVRE. In like manner it follows from the last

$$\text{article that } \frac{1}{m+e} + \frac{1}{m+3e} + \frac{1}{m+5e} + \frac{1}{m+7e} + \&c. \\ = \frac{1}{s-1 \times 2em^s-1} - \frac{se}{12ms+1} + \frac{7s \times s + 1 \times s + 2 \times e^3}{720ms+3} -$$

$\&c.$ For example, if $s = 2$ and $e = \frac{1}{2}$, then $S = \frac{1}{m} - \frac{1}{12m^2}$

$$+ \frac{7}{240m^3} - \frac{31}{1344m^4} + \&c.$$

To compute the sum of the progression $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \&c.$ find the sum of

the terms at the beginning of the series as far as $\frac{1}{m+\frac{1}{2}}$ exclu-

sively, then compute the sum of the subsequent terms by this

theorem. Thus if we add the three first only $1 + \frac{1}{4} + \frac{1}{9} =$

$1.36111 \&c.$ suppose $m + \frac{1}{2} = 4$, or $m = \frac{7}{2}$, and the sum of

the following terms will be $\frac{2}{7} \times 1 - \frac{1}{3 \times 49} + \frac{1}{15 \times 343} - \&c.$

three terms of which series only collected and added to the

former number $1.36111 \&c.$ give $1.64493 \&c.$ for the sum of

the series required, true to the fifth decimal. If $s = \frac{1}{2}$, and $e = \frac{1}{2}$,

$$\text{then } S = \frac{1}{\sqrt{m}} \times 2 - \frac{1}{16mm} + \frac{49}{3072m^3} - \frac{31 \times 11}{32 \times 1024m^5} + \&c.$$

836. These theorems may serve likewise in many cases for

computing the area when the series that arises in the common

method (described above art. 745. $\&c.$) converges at too slow a

rate. For example, let Vmv be a common hyperbola, O the

center, $OR = m$, $Or = n$, and Rr be divided into any even

number of equal parts of which RA is the first and ar the last;

let $RA = e$, and S denote the sum of the ordinates $AF, BE,$

$\dots af$ that insist upon the base at the distance $2e$ from each o-
ther

ther. Then the area $RVvr = 2eS + \frac{2e^2}{12} \times \frac{1}{mm} - \frac{1}{nn} - \frac{2 \times 2 \times 3 \times 7e^4}{720} \times \frac{1}{m^4} - \frac{1}{n^4} + \frac{2 \times 2 \times 3 \times 4 \times 5 \times 31e^6}{30240} \times \frac{1}{m^6} - \frac{1}{n^6} - \&c.$ because in this case $y = \frac{1}{x}$, $\dot{y} = -\frac{x}{x^2}$, $\ddot{y} = -\frac{6x^3}{x^4}$, $\&c.$ Hence, if $OR = m = 1$, $Or = n = 2$, and $RA = e = \frac{1}{8}$, then $S = \frac{8}{9} + \frac{8}{11} + \frac{8}{13} + \frac{8}{15}$, and $2eS = \frac{2}{9} + \frac{2}{11} + \frac{2}{13} + \frac{2}{15}$; $\frac{2e^2}{12} \times \frac{1}{mm} - \frac{1}{nn} = \frac{1}{512}$; $\frac{2 \times 2 \times 3 \times 7e^4}{720} \times \frac{1}{m^4} - \frac{1}{n^4} = \frac{7}{64 \times 64 \times 64}$; and by so few sub-divisions of the base Rr

and terms of the series, the area $RrvV$, or the hyperbolic logarithm of 2, is 0.693146 &c. which differs from the truth by an unit only in the sixth decimal.

837. The logarithm of m being given, the logarithm of $m+z$ is assigned by this theorem, $\log. \frac{m+z}{m} = \log. m+z - \log. m = \frac{2z}{2m+z}$ multiplied by the series $1 - \frac{z}{12} \times \frac{1}{m+z} - \frac{1}{m} + \frac{z^3}{360} \times \frac{1}{m+z^3} - \frac{1}{m^3} - \frac{z^5}{1260} \times \frac{1}{m+z^5} - \frac{1}{m^5} + \&c.$ For let $EA = m$, *Fig. 316.* $Aa = z$, FMf the logarithmic curve having its asymptote EZ perpendicular to EA , $EP = x$, $PM = y$, the *modulus* or ordinate $Ee = 1$; then by the nature of the figure (art. 178.) $\dot{y} = \frac{x}{x}$, or $xy = x$; consequently, MN being perpendicular to the asymptote in N , the area $EeFMN = x - 1$, $eMP = EP \times PM = EeFMN = x \times \log. x - x + 1$, and the area $AFfa = \frac{m+z}{z} \times \log. \frac{m+z}{m} - m \times \log. m - z = A$. And because $a = af - AF = \log. \frac{m+z}{m} - \log. m$, (supposing $z = 1$), $AF = a = \log. m =$ (art. 829.) $\frac{A}{z} - \frac{a}{2} + \frac{z\beta}{12} - \frac{z^3\delta}{720} + \frac{z^5\zeta}{30240} - \&c. = \frac{m}{z} + \frac{1}{2} \times \log. \frac{m+z}{m} - \frac{m}{z} - \frac{1}{2} \times \log. m - 1 + \frac{z\beta}{12} - \frac{z^3\delta}{720} + \frac{z^5\zeta}{30240} - \&c.$ Therefore $\frac{m}{z} + \frac{1}{2} \times \log. \frac{m+z}{m} - \log. m = 1 - \frac{z\beta}{12}$

$\frac{z\beta}{12} + \frac{z^3\delta}{720} - \frac{z^5\zeta}{30240} + \&c. =$ (because $y = \frac{x}{m+z}$ and $\beta = \frac{1}{m+z}$
 $- \frac{1}{m}$, $\dot{y} = \frac{2x}{x^2}$ and $\delta = \frac{2}{m+z} - \frac{2}{m^2}$, $\&c.$) $1 - \frac{z}{12} \times$
 $\frac{1}{m+z} - \frac{1}{m} + \frac{z^3}{360} \times \frac{1}{m+z} - \frac{1}{m^3} - \frac{z^5}{1260} \times \frac{1}{m+z} - \frac{1}{m^5}$. Hence
 if we suppose $m=1$, and $z=1$, because $\log. 1 = 0$, it follows
 that $\frac{3}{2} \times \log. 2 = 1 + \frac{1}{12 \times 2} - \frac{7}{360 \times 8} + \frac{31}{1260 \times 32} - \&c.$

And by supposing $z = \frac{1}{2}$, $\frac{5}{2} \times \log. \frac{3}{2} = 1 + \frac{1}{72} - \frac{19}{360 \times 8 \times 27}$
 $+ \frac{311}{1260 \times 32 \times 343} - \&c.$ By a similar computation, it appears

that if z denote the excess of the logarithm of $a+d$ above
 the logarithm of a , or measure the ratio of $a+d$ to a , then d
 the difference of the numbers may be found by dividing az by
 the series $1 - \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{720} + \frac{z^6}{30240} - \&c.$ Other the-
 orems of this kind may be derived from art. 832.

838. Let it be required to find the sum of the logarithms of
 a series of numbers $m+e, m+3e, m+5e, m+7e, \dots$
 $n-e$, in arithmetical progression, where $m+e$ denotes the least
 term, $n-e$ the greatest, and $2e$ the common difference of the
 terms; or to find the logarithm of the product $\overline{m+e} \times \overline{m+3e} \times$
 $\overline{m+5e} \times \overline{m+7e} \times \dots \times \overline{n-e}$, when all these numbers are
 supposed to be multiplied continually by one another. For this
 end, the figure being the same as in the last article, let EA be
 now equal to $m+e$, $Ea = n-e$, take AR from A towards E
 equal to e , and ar the contrary way equal to AR , and still sup-
 pose the fluxion of the base equal to 1; then $ER = m$, $Er = n$,
 the area $RVvr = n \times \log. n - m \times \log. m - n + m = A$,
 the difference of the fluxions of the ordinates rv and RV is

$$\begin{aligned}
 \frac{1}{n} - \frac{1}{m} &= \beta, \delta = \frac{2}{n^3} - \frac{2}{m^3}, \zeta = \frac{24}{n^5} - \frac{24}{m^5}, \theta = \frac{720}{n^7} \\
 &- \frac{720}{m^7}, \&c. \text{ Therefore (by art. 832.) } S = \frac{A}{2e} - \frac{e\beta}{12} + \frac{7e^3\delta}{720} \\
 &- \frac{31e^5\zeta}{30240} + \&c. = \frac{n \times \log. n - m \times \log. m}{2e} - \frac{n-m}{2e} - \frac{e}{12} \times
 \end{aligned}$$

$$\times \frac{1}{n} - \frac{1}{m} + \frac{7e^3}{360} \times \frac{1}{n^3} - \frac{1}{m^3} - \frac{31e^5}{1260} \times \frac{1}{n^5} - \frac{1}{m^5} + \frac{127e^7}{1680} \times \frac{1}{n^7} - \frac{1}{m^7} - \&c.$$

And this is the same solution which Mr. STIRLING derives from his method, *prop. 28. de interpol. serierum.*

839. The terms in arithmetical progression being represented by $m, m+e, m+2e, m+3e, m+4e, \dots, n-e$, where m denotes the least term, e the common difference, and $n-e$ the greatest term; the sum of the logarithms computed by the theorem for

S , in art. 830. is equal to the excess of the series $\frac{n}{e} - \frac{1}{2} \times \log. n$

$$- \frac{n}{e} + \frac{e}{12n} - \frac{e^3}{360n^3} + \frac{e^5}{1260n^5} - \&c. \text{ above } \frac{m}{e} - \frac{1}{2} \times$$

$$\log. m - \frac{m}{e} + \frac{e}{12m} - \frac{e^3}{360m^3} + \frac{e^5}{1260m^5} - \&c. \text{ For if}$$

we now suppose $EA = m$, and $Ea = n$, AF will be the first ordinate, the area $AFfa = n \times \log. n - m \times \log. m$, $a = af - AF = \log. n - \log. m$, the difference of the fluxions of af and AF or β

$$= \frac{1}{n} - \frac{1}{m}, \delta = \frac{2}{n^2} - \frac{2}{m^2}, \&c. \text{ and the theorem appears}$$

by substituting these values for $A, \beta, \delta, \&c.$ in the equation for S in art. 830. This coincides with the value of S derived by Mr. DE MOIVRE in a different manner *Suppl. ad miscel. Analyt.*

840. The sum of the logarithms of the odd numbers, 3, 5, 7, 9, 11, $\dots, n-1$ is obtained expeditiously, when n is a large number, by computing $\frac{n}{2} \times \log. n - \frac{n}{2} - \frac{1}{12n} + \frac{7}{360n^3}$

$$- \frac{31}{1260n^5} + \frac{127}{1680n^7} - \&c. \text{ and thereafter adding } \frac{\log. 2}{2}$$

or the constant logarithm .346573590 &c. Because if we suppose in art. 838. $e = 1$, and $m = 2$, then $\frac{m \times \log. m}{2} - \frac{m}{2} - \frac{1}{12m}$

$$+ \frac{7}{360m^3} - \frac{31}{1260m^5} + \&c. = \log. 2 - 1 - \frac{1}{12 \times 2} + \frac{7}{360 \times 8}$$

$$- \frac{31}{1260 \times 32} + \&c. = (\text{by art. 837.}) \log. 2 - \frac{3}{2} \times \log. 2 =$$

$-\frac{1}{2} \times \log. 2$; and this quantity is to be subtracted from $\frac{n}{2} \times \log. n - \frac{n}{2} - \frac{1}{12n} + \&c.$ in order to obtain the sum of the logarithms of 3, 5, 7, . . . $n - 1$ by art. 838.

841. The sum of a series, of which the terms are alternately positive and negative, is found by computing separately the sums of such as are affected with the same sign by either of the theorems in art. 830. or 832. and then taking the difference of these sums. But when the terms, which are added and subtracted alternately, may be considered as the successive ordinates of the same figure, the computation of the area may be avoided, and the sum of the series more elegantly obtained by the following theorem. Let AF represent the first positive term, af the term which when the progression is continued succeeds after the last negative term, e the common distance of the ordinates, S the sum of the terms that precede af , and let $\beta, \delta, \zeta, \&c.$ denote the differences by which the alternate fluxions of af exceed the respective fluxions of AF as formerly. Then $S = \frac{AF - af}{2} + \frac{e\beta}{4} - \frac{e^3\delta}{48} + \frac{e^5\zeta}{480} - \&c.$ For the sum of the positive terms (by art. 830. the common distance of the ordinates which represent them being $2e$), is $\frac{A}{2e} - \frac{a}{2} + \frac{2e\beta}{12} - \frac{8e^3\delta}{720} + \&c.$ and the sum of the negative terms is (art. 832.) $\frac{A}{2e} - \frac{e^3}{12} + \frac{7e^5\delta}{720} - \&c.$ the difference of which (a being equal to $af - AF$) is $\frac{AF - af}{2} + \frac{e\beta}{4} - \frac{e^3\delta}{48} + \&c.$ If the first, third and higher alternate fluxions of af vanish, and $\beta, \delta, \zeta, \&c.$ represent the first, third, and higher alternate fluxions of AF , without changing their signs, then $S = \frac{AF - af}{2} - \frac{e\beta}{4} + \frac{e^3\delta}{48} - \frac{e^5\zeta}{480} + \frac{17e^7\theta}{80640} - \&c.$

842. Hence if $EA = 2$, $AF = \log. 2$, $Ea = n$, $af = \log. n$, and $\beta, \delta, \zeta, \&c.$ denote the several fluxions of AF , the logarithm

arithm of the ultimate value of the product $\frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \frac{8}{9} \times \frac{10}{11}$
 $\times \dots \times \frac{n-2}{n-1} \times 2\sqrt{n}$ will be equal to $\frac{\log. 2 - \log. n}{2} - \frac{\beta}{4}$
 $+ \frac{\delta}{48} - \frac{\zeta}{480} + \&c. + \log. 2 + \frac{\log. n}{2} = \frac{3}{2} \times \log. 2 - \frac{\beta}{4}$
 $+ \frac{\delta}{48} - \frac{\zeta}{480} + \&c. =$ (because by art. 837. $\frac{3}{2} \times \log. 2 = 1$
 $+ \frac{\beta}{12} - \frac{7\delta}{720} + \frac{31\zeta}{30240} - \&c.) 1 - \frac{2\beta}{12} + \frac{8\delta}{720} - \frac{32\zeta}{30240}$
 $+ \&c. =$ (because $\beta = \frac{1}{2}, \delta = \frac{1}{8}, \zeta = \frac{1}{32}, \&c.) 1 - \frac{1}{12} + \frac{1}{360} - \frac{1}{1260}$
 $+ \frac{1}{1680} - \&c.$ But (by what has been shewn by Dr. WAL-
 LIS,) if c denote the circumference of the radius unit, $c = 8 \times$
 $\frac{8}{9} \times \frac{24}{25} \times \frac{48}{49} \times \frac{80}{81} \times \&c.$ which product continued till the de-
 nominator of the last fraction be $n-1$ may be expressed by
 $4 \times \frac{4}{9} \times \frac{16}{25} \times \frac{36}{49} \times \frac{64}{81} \times \dots \times \frac{n-2}{n-1} \times n$; consequently $\sqrt{c}$
 is the ultimate value of $\frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \frac{8}{9} \times \dots \times \frac{n-2}{n-1} \times$
 $2\sqrt{n}$; and $\log. \sqrt{c} = \frac{\log. c}{2} = 1 - \frac{1}{12} + \frac{1}{360} - \frac{1}{1260} +$
 $\frac{1}{1680} - \&c.$ This (which was first observed by Mr. STIR-
 LING) serves for abridging the computation in finding the sum
 of the logarithms of the numbers 1, 2, 3, 4, 5, ... $n-1$. For,
 suppose in art. 839. $m = e = 1$, then the latter series in that ar-
 ticle $\frac{m}{e} - \frac{1}{2} \times \log. m - \frac{m}{e} + \frac{e}{12m} - \frac{e^3}{360m^3} + \&c. = -1$
 $+ \frac{1}{12} - \frac{1}{360} + \frac{1}{1260} - \&c. = -\frac{\log. c}{2}$, consequently $S =$
 $\frac{n-1}{2} \times \log. n - n + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \&c. + \frac{\log. c}{2}$
 or if $n-1$ denote the greatest number in the progression, then (sub-
 R r r r 2 titut-

stituting $\frac{1}{2}$ for e in art. 838.) $S = n \times \log. n - n - \frac{1}{24n} + \frac{7}{2880n^3} - \frac{31}{40320n^5} + \&c. + \frac{\log. c}{2}$; which are the rules given for this case in the treatises above mentioned. But if it is required to find the value of $8 \times \frac{8}{9} \times \frac{24}{25} \times \frac{48}{49} \times \&c.$ by the theorem in art. 841 (that is, to compute c from Dr. WALLIS's proposition,) then, because the series for the logarithm of $\sqrt{c}$ converges at too slow a rate, when EA is supposed equal to 2, let r be any greater even number; find the number whose logarithm is $\frac{1}{4r} - \frac{1}{24r^3} + \frac{1}{20r^5} - \&c.$ call this number N , and $\sqrt{c} = 2 \times \frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \frac{8}{9} \dots \times \frac{r-2}{r-1} \times \frac{\sqrt{r}}{N}$. If $r=10$, then let N be the number whose logarithm is equal to $\frac{1}{40} - \frac{1}{24000} + \frac{1}{2000000} - \&c.$ and $\sqrt{c} = \frac{256}{315} \times \frac{\sqrt{10}}{N}$.

843 In like manner the logarithm of the ultimate value of other products of this kind may be found; as of $3 \times \frac{21}{25} \times \frac{77}{81} \times \frac{165}{169} \times \frac{285}{289} \times \&c.$ where the denominators are the squares of the odd numbers 5, 9, 13, 17, 21, &c. whose common difference is 4, and each numerator is less than its denominator by 4. Let the ultimate value of this product be called p and $\sqrt{p}$ will be the ultimate value of $\frac{3}{5} \times \frac{7}{9} \times \frac{11}{13} \times \frac{15}{17} \times \dots \times \frac{n-4}{n-2} \times \sqrt{n}$. Let r be any number in the progression, 3, 7, 11, 15, 21, 25, &c. and N the number whose logarithm is equal to $\frac{1}{2r} - \frac{1}{3r^3} + \frac{8}{8r^5} - \&c.$ then $\sqrt{p} = \frac{\sqrt{r}}{N} \times \frac{3}{5} \times \frac{7}{9} \times \frac{11}{13} \times \frac{15}{17} \dots \times \frac{r-4}{r-2}$.

844. The problem, concerning the ratio of the sum of all the *uncia* of the power of a binomial to the *uncia* of the middle term, may be resolved by art. 838 or 839, with article 842. or rather by the following theorem. Let r be the exponent of

of the power to which the binomial is to be raised when the exponent is an even number, or equal to this exponent diminished by unit when it is an odd number; and c denote the circumference of the circle when the radius is unit; let N denote the number whose logarithm is equal to $\frac{1}{4 \times r+1}$ —

$\frac{1}{24 \times r+1} + \frac{1}{20 \times r+1} - \&c.$ and the ratio required will be that of $\frac{\sqrt{c \times r+1}}{2N}$ to unit. For this ratio is equal to $1 \times \frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \frac{8}{9} \times \dots \times \frac{r-2}{r-1} \times r$, which (by art. 842.) is equal to the ultimate value of $\frac{\sqrt{c}}{2} \times \frac{r+1}{r+2} \times \frac{r+3}{r+4} \times \frac{r+5}{r+6} \times$

$\dots \times \sqrt{r+s}$ where s is supposed to represent a number that continually increases by the increment 2; and the logarithm of this ultimate value is (by art. 841. supposing $AF = \log. r+1$, and $af = \log. r+s$) $\frac{\log. \sqrt{c}}{2} - \frac{\log. r+s}{2} + \frac{\log. r+1}{2} -$

$\frac{1}{4 \times r+1} + \frac{1}{24 \times r+1} - \frac{1}{20 \times r+1} + \&c. + \frac{\log. r+s}{2} =$

$\log. \frac{\sqrt{c \times r+1}}{2} - \frac{1}{4 \times r+1} + \frac{1}{24 \times r+1} - \frac{1}{20 \times r+1} + \&c.$

$= \log. \frac{\sqrt{c \times r+1}}{2N}$. These are always supposed to be hyperbolic logarithms, but are converted into tabular logarithms, by

multiplying by 0.4342944819 &c. The resolution of this problem derived from other principles may be found in Mr. DE MOIVRE'S *suppl. Miscel. Analyt.* p. 17. and Mr. STIRLING'S *tract. de summat. serier.* p. 119. Because the other coefficients of the terms of a binomial (when the exponent r is an even number) are found by multiplying the coefficient of the middle term by $\frac{r}{r+2} \times \frac{r-2}{r+4} \times \frac{r-4}{r+6} \times \&c.$ these may be likewise found by art. 838. &c. For the use of the properties of the terms of a binomial, when raised to a high power, see Mr. JAMES

JAMES BERNOUILLI'S *ars conject.* part. 4. ch. 4. and Mr. DE MOIVRE'S *doctr. of chances.*

845. The sum of the series $\frac{1}{mr} - \frac{1}{m+e}r + \frac{1}{m+2e}r - \frac{1}{m+3e}r + \frac{1}{m+5e}r - \&c.$ is (by art. 841. because *af* with all its fluxions ultimately vanish in this case) $\frac{1}{2mr} + \frac{reA}{2m} - \frac{r+1 \times r+2}{12mm} \times eeB + \frac{r+3 \times r+4}{10mm} \times eeC - \frac{17r+5 \times r+6}{168mm} \times eeD + \&c.$ where A in the usual manner denotes the first term, $\frac{1}{2mr}$, B the second, C the third not including its sign, &c. If

$$r=1, \text{ then } S = \frac{1}{2m} + \frac{1}{4mm} - \frac{1}{8m^3} + \frac{1}{4m^5} - \frac{17}{8m^7} + \&c.$$

And hence the sum of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \&c.$ (which is equal to the hyperbolic logarithm of 2.) may be easily computed to a great number of decimal places, by first collecting the sum of the terms at the beginning of the series (by common Arithmetic) that preceed $\frac{1}{m}$, so as that *m* may be a pretty large number (equal to 25 or 27 for example,) and then computing the sum of the other terms by this series. If

$$r=2, \text{ then } S = \frac{1}{2mm} + \frac{1}{2m^3} - \frac{1}{2m^5} + \frac{3}{2m^7} - \&c. \text{ whence}$$

the sum of the series $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \&c.$ may be computed in like manner. By supposing $r=1$ and $e=2$, the sum of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \&c.$ may be computed by first collecting the sum of the terms at the beginning of the series that preceed $\frac{1}{m}$, viz.

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \&c. = \frac{1}{m-2}, \text{ by common Arithmetic; and then}$$

$$\text{adding } \frac{1}{2m} + \frac{1}{2mm} - \frac{1}{m^4} + \frac{8}{m^6} - \frac{34}{m^8} + \&c. \text{ This series is}$$

equal to $\frac{1}{8}$ of the circumference of the circle, the radius being equal to 1, by art. 746. and hence the ratio of the circumference

rence to the diameter may be computed to many decimal places with little labour.

846. The theorems in art. 830. and 832. may be applied for approximating to the sum of the series that is formed by substituting successively any numbers in arithmetical progression in

the place of x in the fraction $\frac{1}{x+a \times x+b \times x+c \times \&c.}$ (where $a, b, c, \&c$ represent any given numbers) by what was shewn in the last chapter concerning the area, when the ordinate is equal to such a fraction, &c. And in some cases this sum may be assigned accurately by art. 361.

847. If N represent the number whose hyperbolic logarithm is e , the sum of the series $1 + \frac{e}{2} + \frac{e^2}{12} - \frac{e^4}{720} + \frac{e^6}{30240} - \frac{e^8}{1209600} + \&c. = \frac{eN}{N-1}$: And the sum of the series $\frac{1}{2} - \frac{e^2}{12} + \frac{7e^4}{720} - \frac{31e^6}{30240} + \frac{127e^8}{1209600} - \&c. = \frac{eN}{NN-1}$. These appear by supposing in art. 830. and 832. the curve FMf to be the logarithmic, Aa its asymptote, AF or RV equal to the modulus, and finding the sum of the ordinates by the common rule for a geometrical progression, and putting this sum equal to the value of S in those articles.

848. The base Aa being divided into any number of equal parts represented by n , let the area $AFfa = Q$, the sum of the extreme ordinates $AF + af = A$, the sum of all the intermediate ordinates $BE + CK + \&c. = B$, the base $Aa = R$, and the same quantities be represented by $\beta, \delta, \zeta, \&c.$ as formerly; then

the area $AFfa = Q = \frac{A}{2n+2} + \frac{nB}{nn-1} \times R - \frac{R^3 \delta}{720nn} + \frac{R^5 \zeta}{30240n^3} \times \frac{nn+1}{n^2} - \&c.$ For, supposing in art. 830. $e = \frac{R}{n}$, $S + \frac{af - AF}{2} = B + \frac{A}{2} = \frac{nQ}{R} + \frac{R\beta}{12n} - \frac{R^3 \delta}{720n^3} + \frac{R^5 \zeta}{30240n^5} - \&c.$ and supposing $e = R$ in the same theorem, $\frac{AF + af}{2} = \frac{A}{2} = \frac{Q}{R} + \frac{R\beta}{12} - \frac{R^3 \delta}{720} + \frac{R^5 \zeta}{30240} - \&c.$ then if we extermin-

nate

nate β by these two equations, the proposition will appear. If we neglect δ , ζ , θ , &c. $AFfa = \frac{AR}{2n+2} + \frac{nBR}{nn-1}$. Suppose $n=2$, or that there are three ordinates only, (in which case B denotes the middle ordinate) then the area $AFfa = \frac{A+4B}{6} \times R - \frac{R^3\delta}{4 \times 720} + \frac{5R^5\zeta}{16 \times 30240} - \&c.$ If we suppose $n=3$, or that there are four ordinates only, B will represent the sum of the second & third, and the area $AFfa = \frac{A+3B}{8} \times R - \frac{R^3\delta}{9 \times 720} + \frac{R^5\zeta}{81 \times 3024} - \&c.$ By neglecting δ , ζ , θ , &c. we shall have two of the theorems given by Sir ISAAC NEWTON and others for computing the area from equidistant ordinates, the latter of which (viz. $AFfa = \frac{A+3B}{8} \times R$) is much recommended by Mr. COTES.

849. By exterminating δ , ζ , θ , &c. successively, other theorems will be found by which the area will be more and more accurately determined from the ordinates. Let there be five ordinates, A the sum of the first and last, B the sum of the second and fourth, and C the middle ordinate; then the area $AFfa = \frac{7A+32B+12C}{90} \times R - \frac{31R^5\zeta}{6 \times 16 \times 16 \times 30240} + \&c.$

For by the rule for three ordinates $\frac{Q}{R} = \frac{A+4C}{6} \times R - \frac{R^3\delta}{4 \times 720} + \frac{5R^5\zeta}{16 \times 30240}$. By dividing the base into two equal parts and computing from the same rule the area that stands up

on each part, and adding these areas together, $\frac{2Q}{R} = \frac{A+4B+2C}{6} - \frac{R^3\delta}{32 \times 720} + \frac{5R^5\zeta}{2 \times 16 \times 16 \times 30240} - \&c.$ then by exterminating

δ by these two equations, the proposition appears. These theorems may be continued in like manner, and some judgment formed of the accuracy of the several rules, by comparing the quantities that are neglected in them.

terme-

850. The theorems in art. 830. and 832. from which we have drawn so many conclusions, may be of use for interpolating the terms of a series likewise, or for finding the intermediate ordinates of a figure when the equidistant primary ordinates are given. When the equation of the figure FMf is known, the intermediate ordinates are found without any difficulty, by substituting the intermediate values of the base in the equation; but it is not so obvious how we are to interpolate the values of S, or the sums of those ordinates. Suppose FNz to be the figure FIG. 317. whose successive ordinates at the points A, B, C, D, &c. are always equal to the successive sums of the ordinates of the figure FMf at the same points beginning with AF; that is, let $AF = AF$, $B\varepsilon = AF + BE$, $C\kappa = B\varepsilon + CK$, $D\lambda = C\kappa + DL$, &c. and let it be required to determine any intermediate ordinate PN of the figure FNz. Let this ordinate PN meet the curve FMf in M, $AF = a$, $PM = y$, the common distance of the ordinates $AB = e$, the area AFMP = Q, $\dot{y} - \dot{a} = \beta$, $\ddot{y} - \ddot{a} = \delta$, &c. then $PN = \frac{Q}{e} + \frac{a+y}{2} + \frac{e\beta}{12} - \frac{e^3\delta}{720} + \frac{e^5\zeta}{30240} - \frac{e^7\theta}{1209600} + \text{\&c.}$ because if we suppose PN to move successively into the places of the ordinates of the figure FNz at A, B, C, D, &c. its successive values will be rightly determined by this theorem, by art. 830. Or if we would avoid the area Q in the theorem for PN, let $AP = m$, and since $\frac{a+y}{2} = \frac{Q}{m} + \frac{m\beta}{12} - \frac{m^3\delta}{720} + \frac{m^5\zeta}{30240} - \text{\&c.}$ it follows that $PN = \frac{a+y}{2} \times \frac{e+m}{e} + \frac{e^2-m^2}{12e} \times \beta - \frac{e^4-m^4}{720e} \times \delta + \frac{e^6-m^6}{30240e} \times \zeta - \text{\&c.}$ A similar theorem follows from art. 832. let AR be taken backwards from A and Pr forwards from P, each equal to $\frac{1}{2} AB$, RV and rv meet FMf in V and v; let Q now denote the area RVvr, and β , δ , ζ , &c. denote the differences by which the first, third, fifth, and higher alternate fluxions of the ordinate rv exceed the respective fluxions of RV, and $AB = e$ as
S f f f
for-

formerly; then any ordinate $PN = \frac{Q}{e} - \frac{e^3}{2 \times 12} + \frac{7e^5}{8 \times 720} - \frac{31e^7}{32 \times 30240} + \&c.$ The ordinates at the points A, B, C, D, &c. are called the primary ordinates of the figures FNz or FMf . If $Pp = AB$, pn meet FNz in n and FMf in m , then $pn = PN + pm$ or $PN - pm$, according as Pp is taken forwards or backwards from P : And hence any intermediate ordinate PN being known, all other ordinates of the figure FNz that are at a distance from it equal to AB , or any multiple of AB , are easily found by adding or subtracting the intermediate ordinates of the figure FMf .

851. Let TX and $T'X'$ be the primary ordinates of the figure FMf adjoining to the intermediate ordinate PN ; bisect TT' in x , let the ordinate xy meet FMf in y , the area $xyvr = q$, the ordinate at T of the figure FNz , viz. $T\tau = f$, $xy = y$, $rv = u$, then $PN = f + \frac{q}{e} - \frac{e}{2 \times 12} \times \overline{u-y} + \frac{7e^3}{8 \times 720} \times \overline{u-y}^3 - \&c.$ For if $RV = a$, the area $RVvr = Q$, then $RVyx = Q - q$, and $PN = \frac{Q}{e} - \frac{e}{2 \times 12} \times \overline{u-a} + \frac{7e^3}{8 \times 720} \times \overline{u-a}^3 - \&c.$ by the last article; and $T\tau = f = \frac{Q-q}{e} - \frac{e}{2 \times 12} \times \overline{y-a} + \frac{7e^3}{8 \times 720} \times \overline{y-a}^3 - \&c.$ by art. 830. consequently $PN - f = \frac{q}{e} - \frac{e}{24} \times \overline{u-y} + \frac{7e^3}{8 \times 720} \times \overline{u-y}^3 - \&c.$ and $PN = f + \frac{q}{e} - \frac{e}{24} \times \overline{u-y} + \frac{7e^3}{8 \times 720} \times \overline{u-y}^3 + \&c.$ This series will converge very fast in many cases, when PN is at a great distance from AF .

852. This leads us to some easy and simple theorems for finding the intermediate terms of a series by interpolating the differences of the terms. First, suppose the differences of the terms to decrease continually, so that by continuing the series these differences may become less than any given quantity, but never vanish; or that the terms of the series being represented by the ordinates of the figure FNz , and their differences by the ordinates of FMf , this latter figure has the base Ff for its asymptote

ptote. In this case πv the term of the series that preceeds the first primary term AF , at any distance $A\pi$ less than AB , is equal to the excess of the sum of the primary differences $AF + BE + CK + \&c.$ above the sum of the interpolated differences $be + ck + dl + \&c.$ the distances $Bb, Cc, Dd, \&c.$ being each equal to $A\pi$, and taken the same way from $B, C, D, \&c.$ For in this case PN is ultimately equal to $T\tau$; that is (supposing $Pr' = \pi A$) $\pi v + be + ck + dl + \&c.$ is ultimately equal to $AF + BE + CK + \&c.$ consequently $\pi v = AF - be + BE - ck + CK - dl + \&c.$

853. For example, suppose $AF = 1, BE = \frac{1}{2}, CK = \frac{1}{3}, DL = \frac{1}{4}, \&c.$ then the successive primary ordinates of the figure FNz will be $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \frac{147}{60}, \&c.$ Let $A\pi = \frac{1}{2} AB$, and because the intermediate differences $be = \frac{2}{3}, ck = \frac{2}{5}, dl = \frac{2}{7}, \&c.$ it follows that $\pi v = 1 - \frac{2}{3} + \frac{1}{2} - \frac{2}{5} + \frac{1}{3} - \frac{2}{7} + \frac{1}{4} - \frac{2}{9} + \&c. = 2 \times \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \&c. =$ (because $\log. 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \&c.$) $2 \times 1 - \log. 2.$ And the other intermediate terms are found by adding successively the intermediate differences $\frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \&c.$ *

854. In like manner, if we suppose $AF = 1, BE = \frac{1}{4}, CK = \frac{1}{9}, DL = \frac{1}{16}, \&c.$ or the successive primary ordinates of FMf to be the squares of $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \&c.$ and we suppose $A\pi = \frac{1}{2} AB$, then the intermediate differences $be, ck, dl, \&c.$ will be $\frac{4}{9}, \frac{4}{25}, \frac{4}{49}, \&c.$ the ordinates $AF, B\epsilon, C\kappa, D\lambda, \&c.$ will be

S f f f 2 1,

* The intermediate terms of this series are determined by the learned Mr. EULER, *Comment. petropol. Tom. 5. p. 93.* by finding a fluent that expresses the terms of the series in a general manner, which in this case is $F, \frac{1-x^n}{1-x} \times x$ supposing n to denote the place of the term in the series (that is, 1 for the first term, 2 for the second, $\&c.$ and $\frac{1}{2}$ for the term πv), and 1 to be substituted for x after the fluent is determined; whence $\pi v = F, \frac{1-\sqrt{x}}{1-x} \times x = 2 - 2 \times \log. 2.$ I take this opportunity to mention, that having occasionally shewn in 1737, the 292, 293. pages of this treatise (after they were printed) to Mr. STIRLING, he took notice that a theorem similar to the first of these described in art. 352. had been communicated to him by Mr. EULER.

1, $1 + \frac{1}{4}$, $1 + \frac{1}{4} + \frac{1}{9}$, $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$, &c. and the ordinate πv that stands before the first primary ordinate AF, at half the common distance of the ordinates, will be equal to $1 - \frac{4}{9} + \frac{1}{4} - \frac{4}{25} + \frac{1}{9} - \frac{4}{49} + \dots = 4 \times \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \dots$. Therefore, if the sum of the series $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{49} + \dots$ (which may be computed easily from art. 845.) be denoted by N, then $\pi v = 4 - 4N$. If $AF = 1$, $BE = \frac{1}{9}$, $CK = \frac{1}{9}$, $DL = \frac{1}{16}$, &c. And $A\pi = \frac{1}{2} AB$, then $\pi v = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$ which is equal to the eight part of the circumference of the radius unit.

FIG. 318.

855. When the terms may be continued without end, and their second differences decrease so as ultimately to vanish, let K denote the ultimate value of the first differences of the terms; and πv will be equal to $\frac{AB - A\pi}{AB} \times K$ added to the excess of the sum of the primary differences $AF + BE + CK, \dots$ above the sum of the intermediate differences $be + ck + dl + \dots$ because in this case the fluxions of rv and xy ultimately vanish, PN is ultimately equal to $f + \frac{q}{e}$, q to $K \times xr = K \times \overline{AB - A\pi}$, and consequently $\pi v = \frac{AB - A\pi}{AB} \times K + AF - be + BE - ck + \dots$. A like theorem may be applied, when the second differences of the terms continually approach to a certain limit.

856. The series 1, 1×1 , 1×2 , $1 \times 2 \times 3$, $1 \times 2 \times 3 \times 4$, $1 \times 2 \times 3 \times 4 \times 5$, &c. being proposed; let it be required to find the term that is betwixt the two first primary terms at equal distances from each. The differences of the logarithms of the terms are $\log. 1$, $\log. 2$, $\log. 3$, $\log. 4$, $\log. 5$, &c. and the ordinates of the figure FMf being supposed to represent these logarithms, the intermediate ordinates will be $\log. \frac{3}{2}$, $\log. \frac{5}{2}$, $\log. \frac{7}{2}$, $\log. \frac{9}{2}$, &c. Therefore the logarithm of the term required is $\frac{K}{2} + \log 1 - \log. \frac{3}{2} + \log. 2 - \log. \frac{5}{2} + \log.$

Fig. 308. N.1.

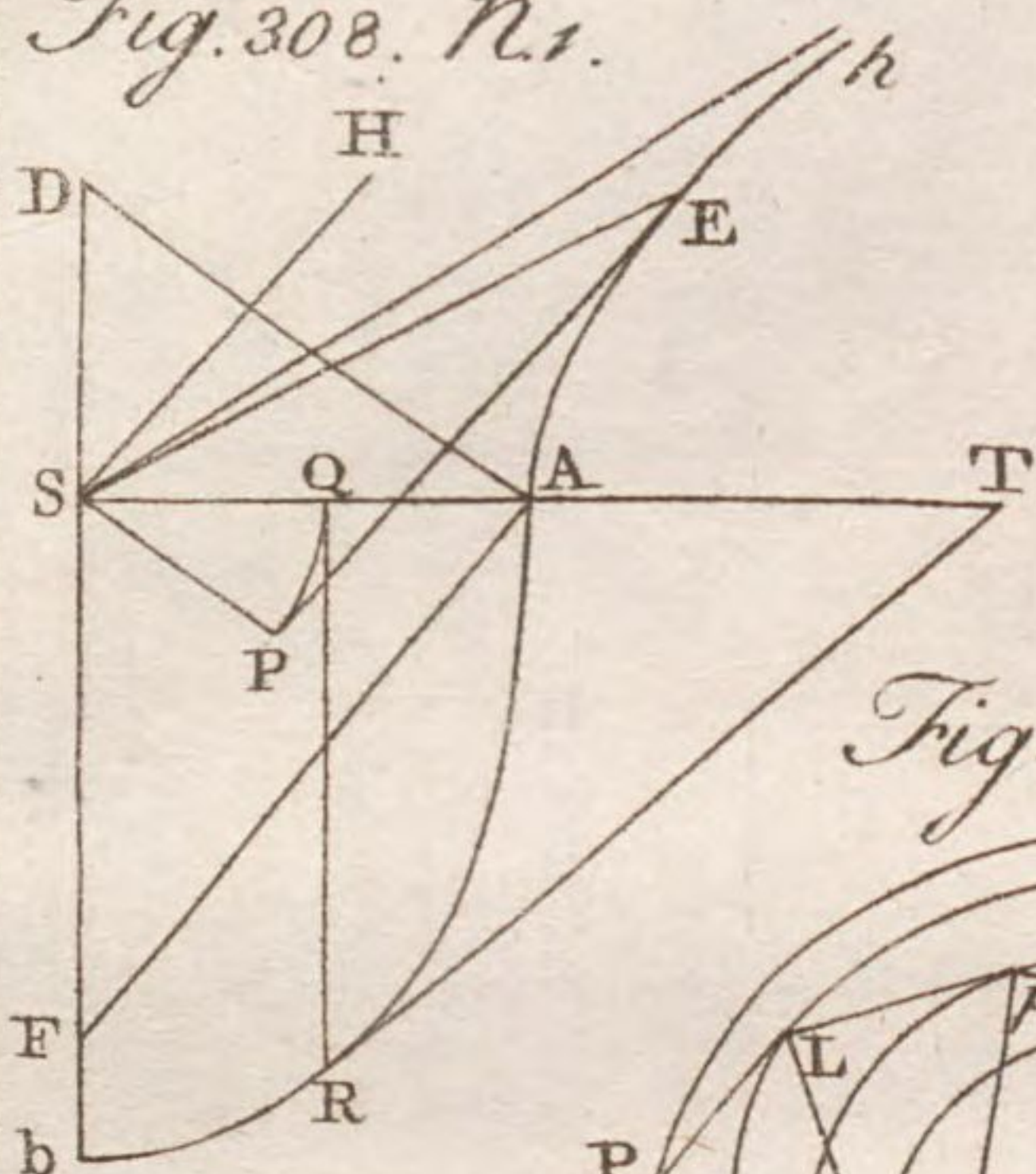


Fig. 308. N.2. Art. 806.

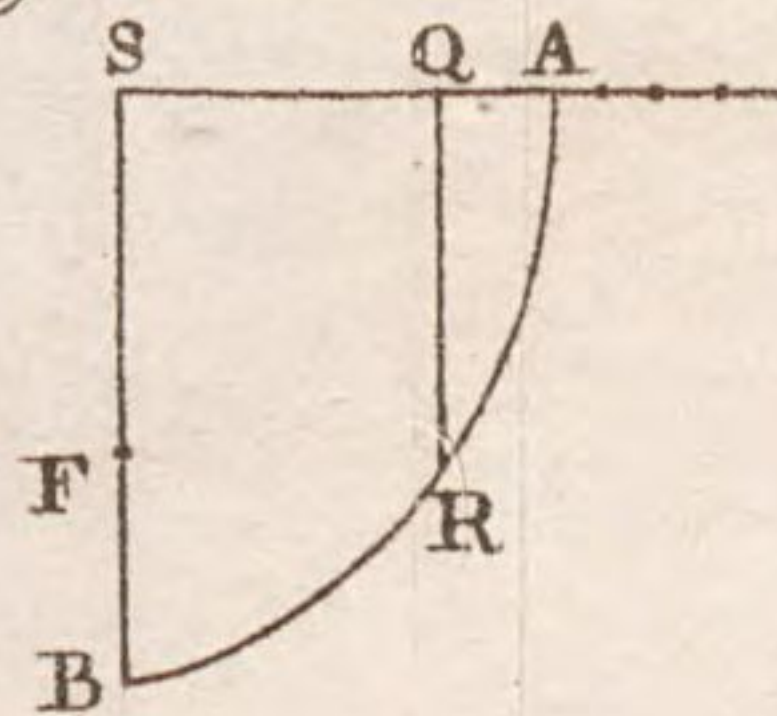


Fig. 309.

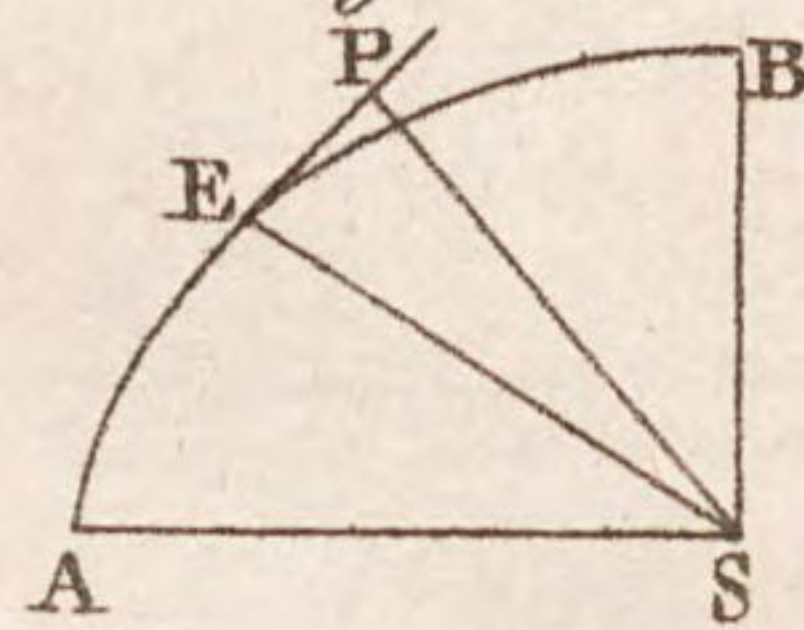
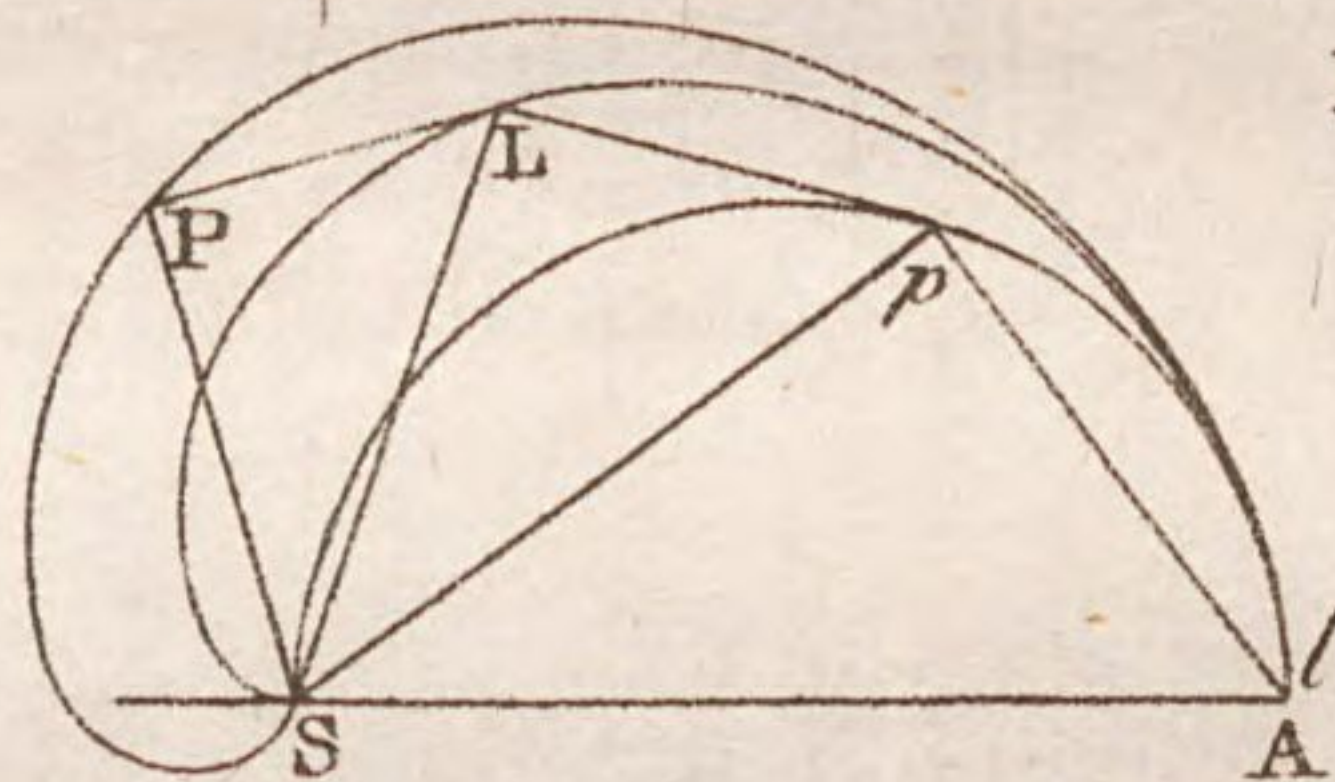


Fig. 311. N.2.



Pl. XXXVI Pa. 692
Fig. 310.

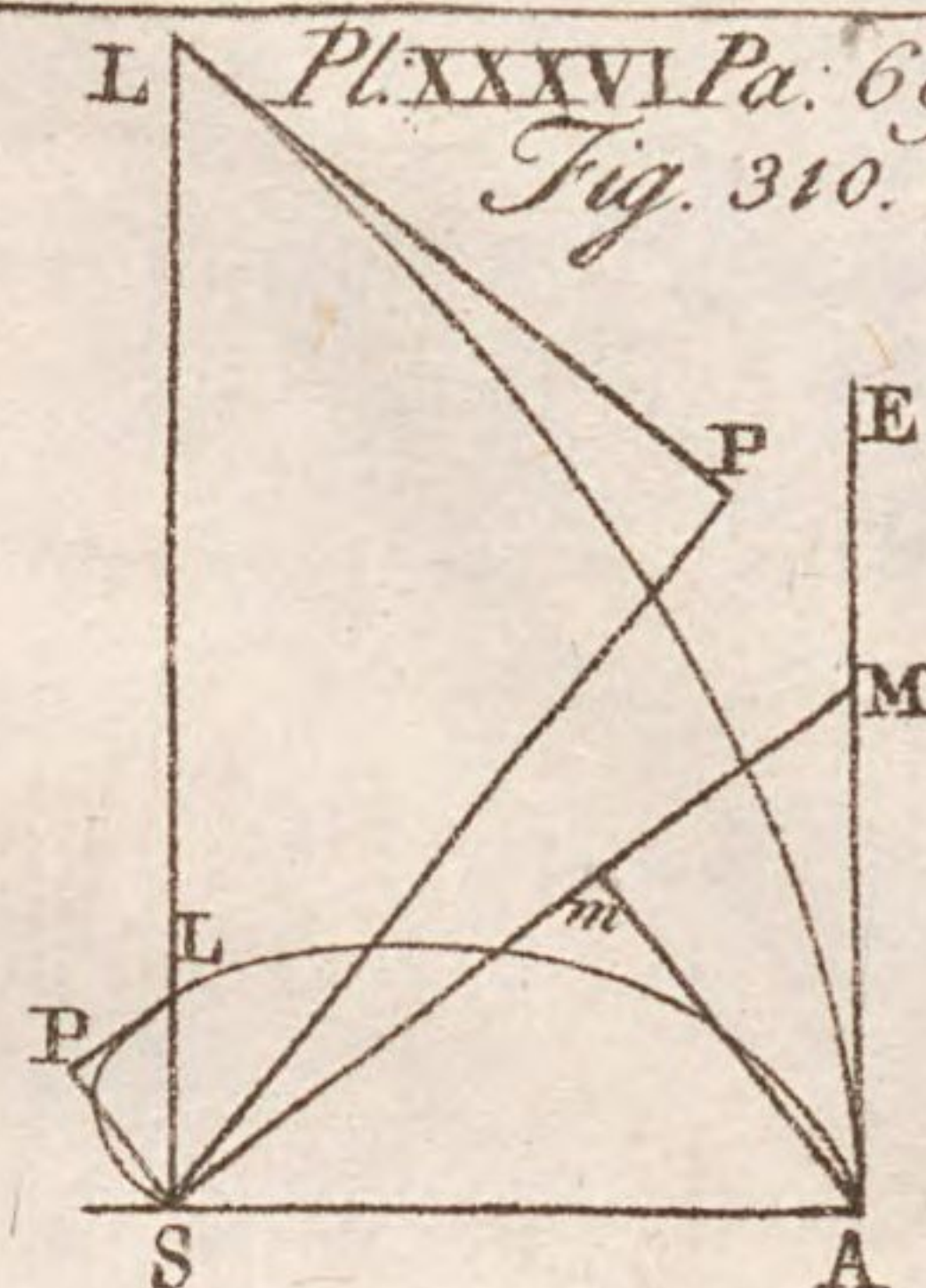


Fig. 311. N.3.

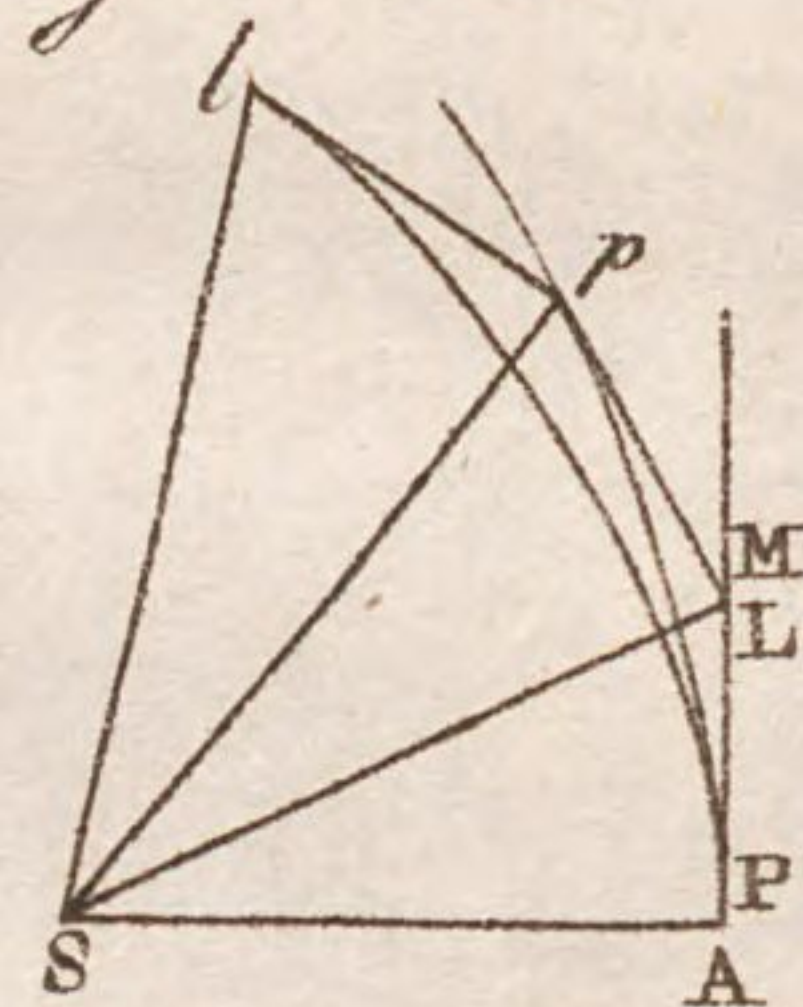


Fig. 312.

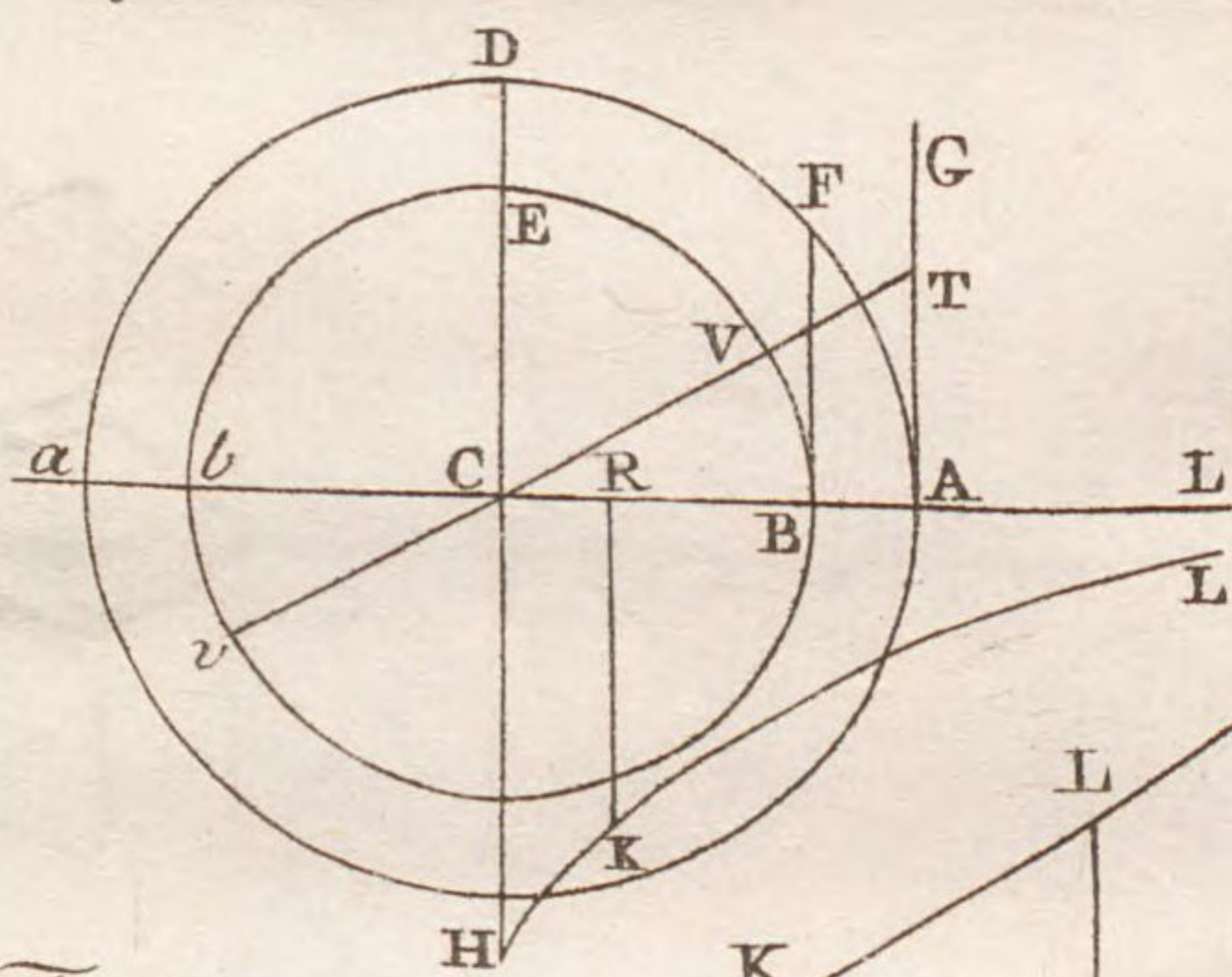


Fig. 314. N.1.

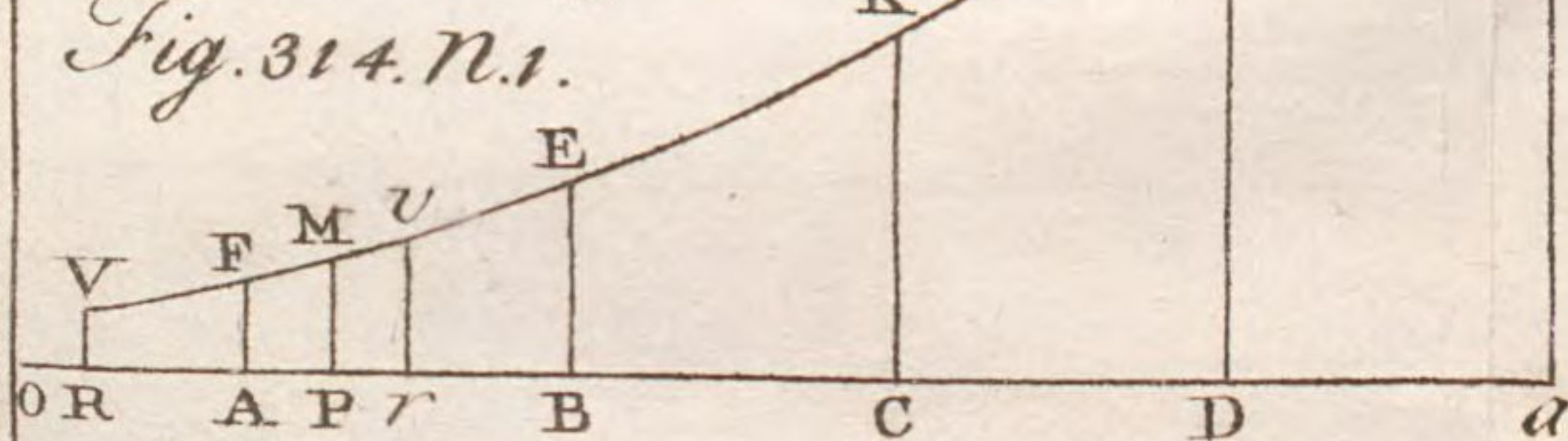


Fig. 316.

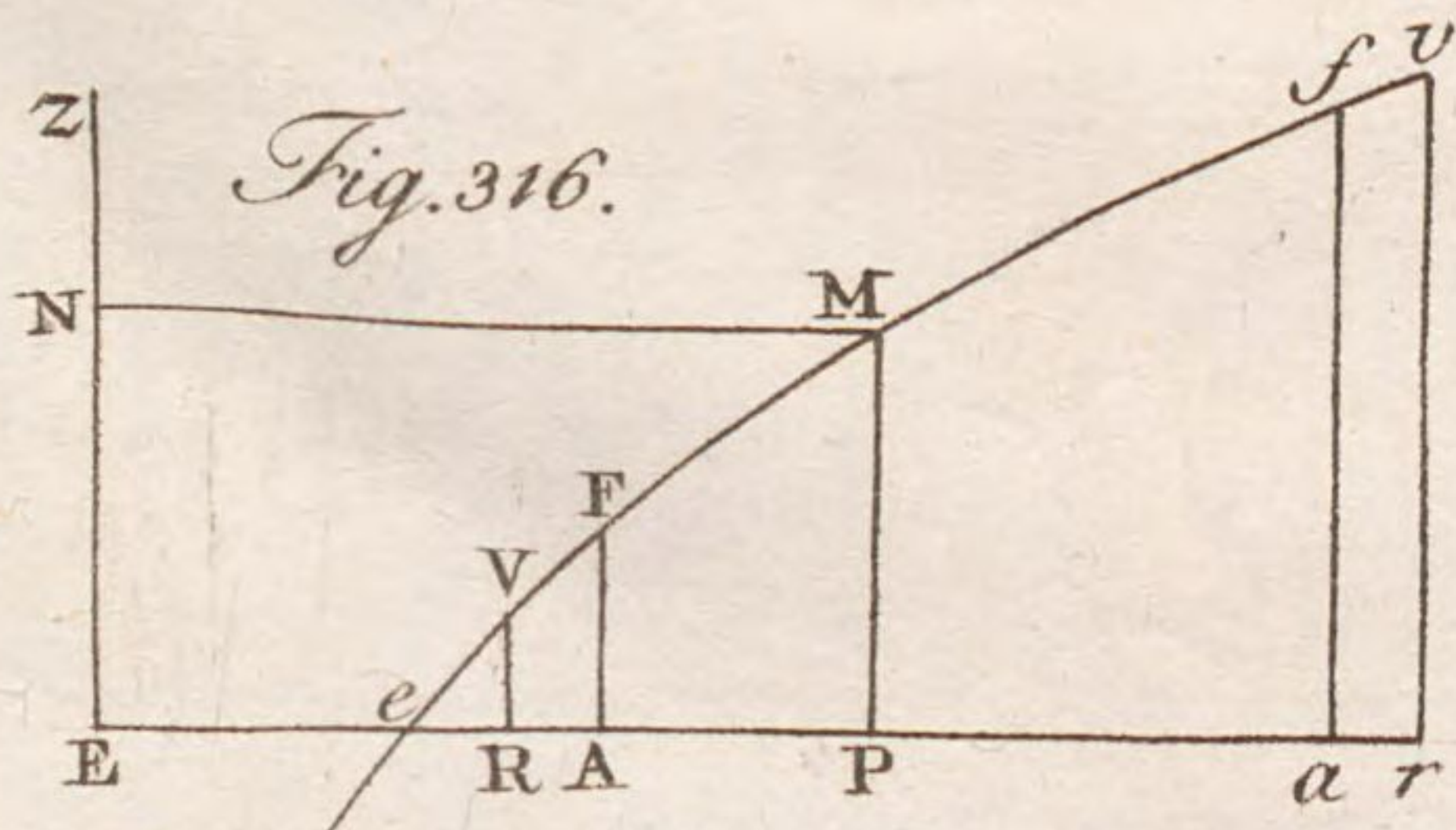


Fig. 313.

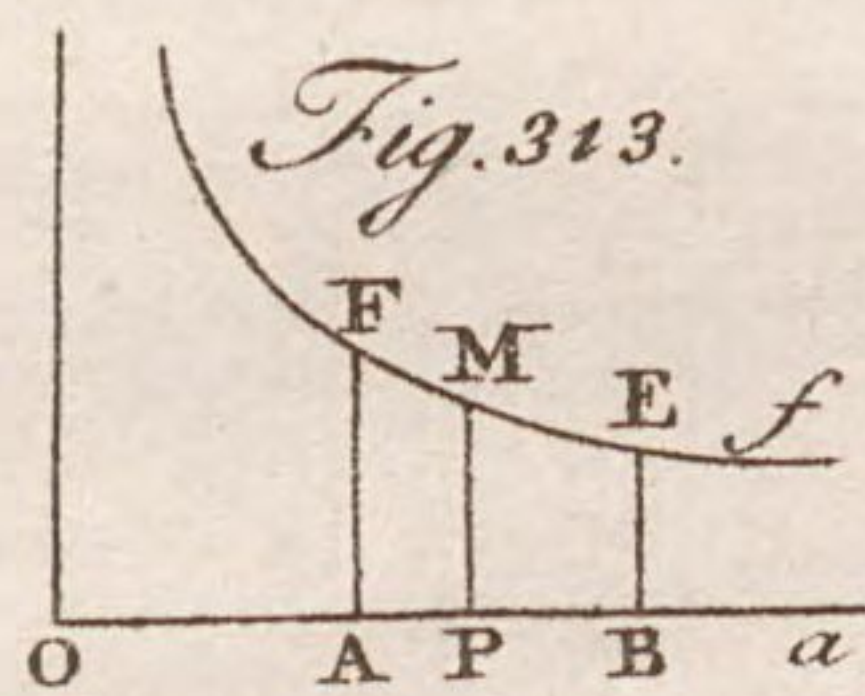


Fig. 315.

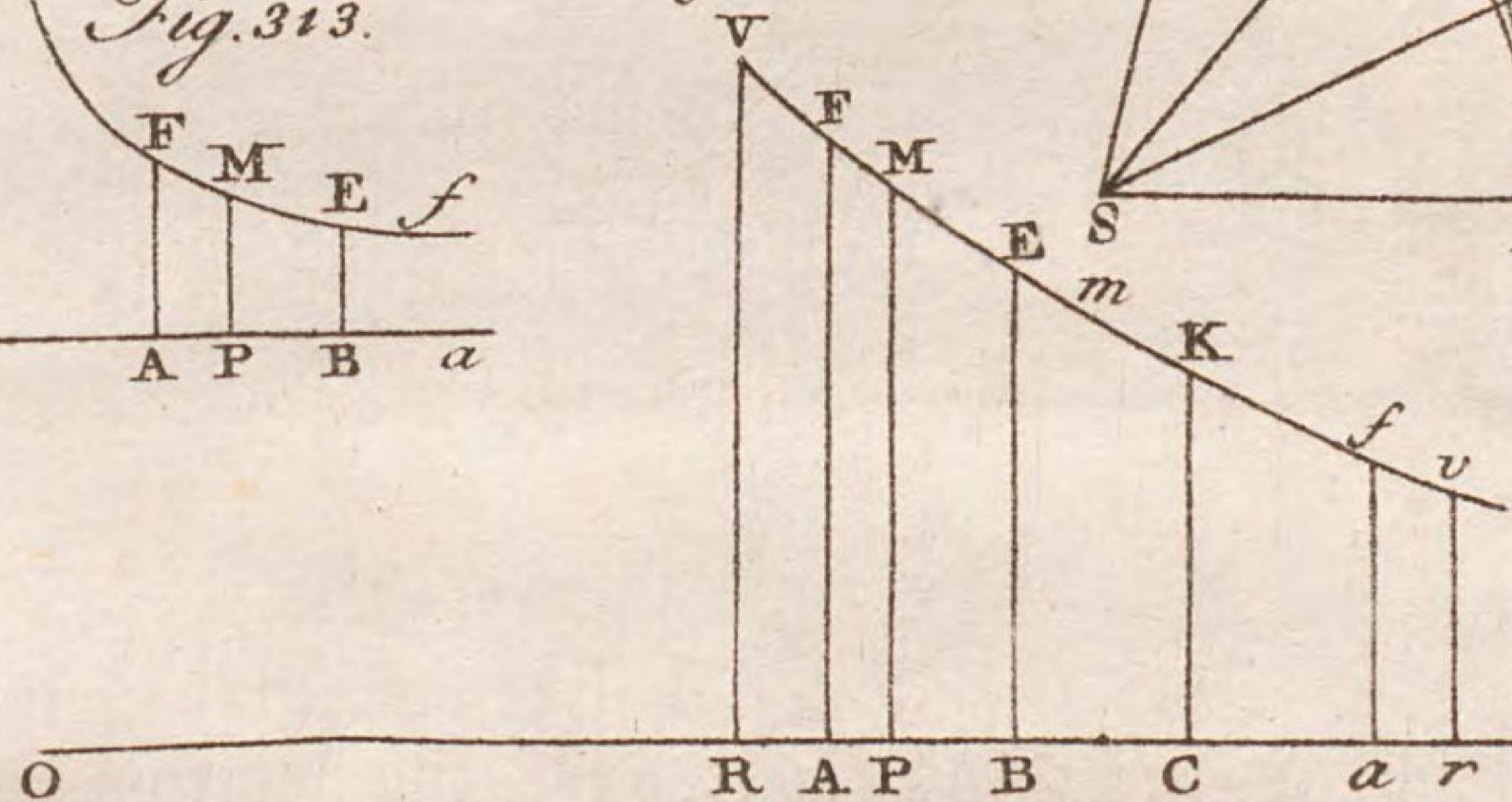


Fig. 314. N.2.

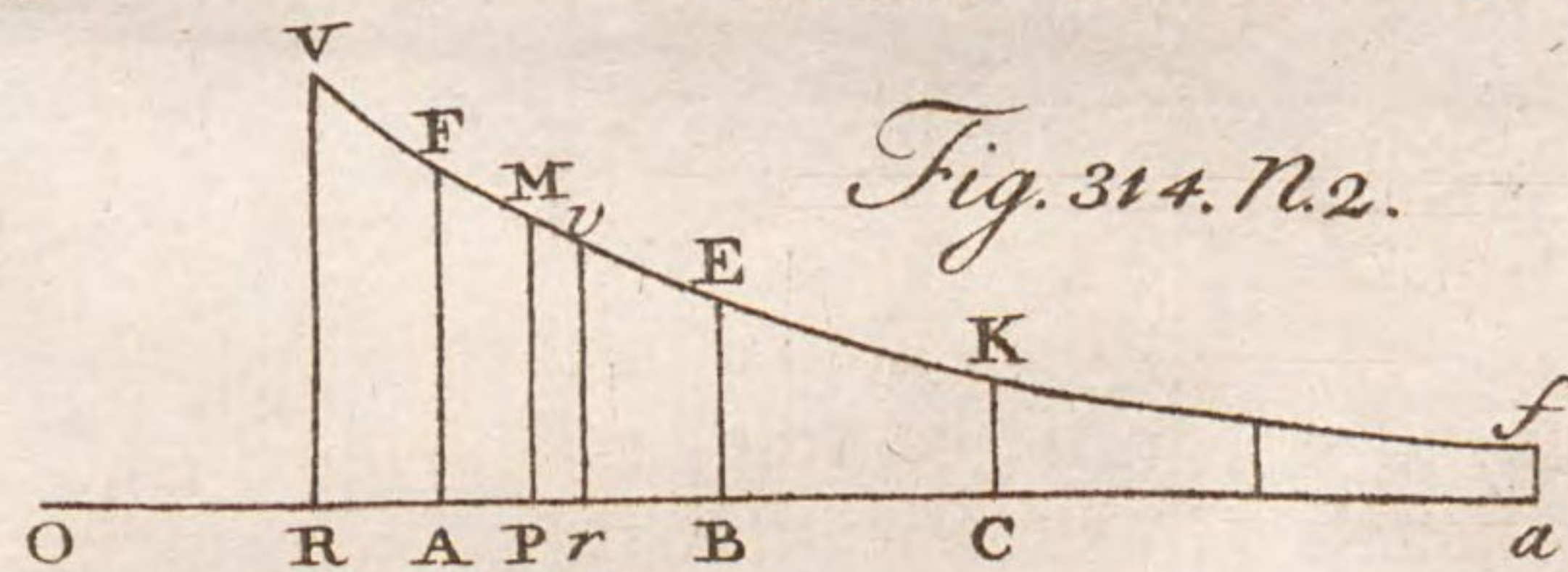
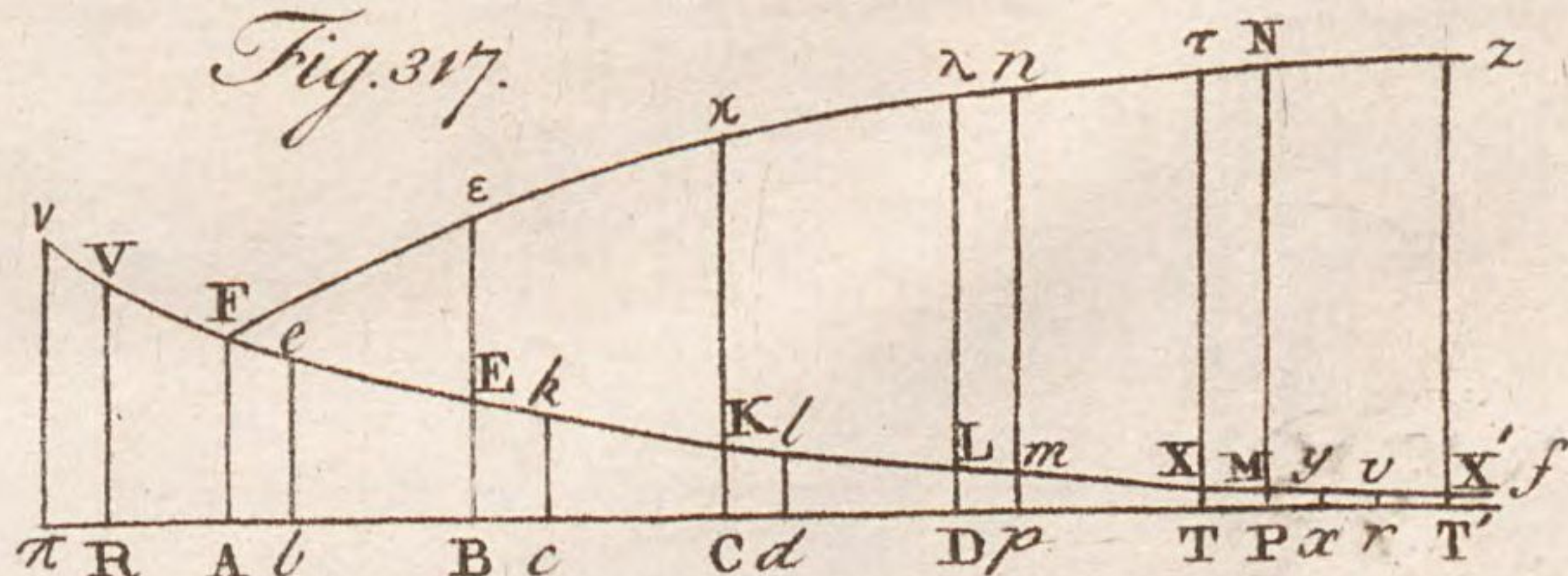


Fig. 317.



$\log. 3 - \log. \frac{7}{2} + \&c.$ which is equal to the logarithm of the ultimate value of $\frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \frac{8}{9} \dots \times \frac{n}{n-1} \times \sqrt{n+1} =$ (by what was shewn in art. 842. from Dr. WALLIS) $\log. \frac{\sqrt{e}}{2}$; consequently the term required is equal to half the square-root of the circumference of the radius 1; which is agreeable to what has been discovered by other methods. This subject might be prosecuted further, and other instances given of the use of the method of fluxions, in finding the sum of a series, or interpolating its terms; but we proceed to what is more necessary for bringing this treatise to a proper conclusion.

CHAP. V.

Of the general rules for the resolution of problems.

857. **I**T remains, that we describe briefly the general rules that FIG. 319. are derived from this method for the resolution of problems, and illustrate them by examples. The base AP being represented by x , and the ordinate PM by y , the *subtangent* PT (which is the right line intercepted upon the base betwixt the ordinate and tangent) is found by computing $\frac{y\dot{x}}{\dot{y}}$. When y increases while x increases, this value of PT is positive, and PT is on the same side of P with PA; but when y decreases while x increases, this value of PT is negative, and PT is on the other side of P. If x vanish in respect of y , PT vanishes, and the ordinate is the tangent; but if y vanish in respect of x , the tangent is parallel to the base. If the curve FM be represented by z , then the tangent $MT = \frac{yz}{\dot{y}} = \frac{y\sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{y}}$. If MN perpendicular to the tangent MT meet the base in N, PN (which is sometimes called the *subnormal*) $= \frac{yy}{x}$. These follow

low from art. 188. &c. by which $\dot{x}$, $\dot{y}$ and $\dot{z}$ are in the same proportion as the right lines PT, PM and MT; or as PM, PN and MN. For example, if $y^m = a^{m-1}x$, then (art. 728.) $\frac{my}{y}$

FIG. 320. $= \frac{\dot{x}}{x}$, and $PT = \frac{y\dot{x}}{y} = m\dot{x} = m \times AP$. Let the ray SE revolve about a given center S, and meet the curve AEB in E, the ark fE be described from the center S, $SE = r$, the ark of the curve $AE = s$, the fluxion of the circular ark fE be represented by $\dot{z}$, and SP be perpendicular from S on the tangent EP in P, then $SP = \frac{r\dot{z}}{s}$, and $EP = \frac{r\dot{r}}{s}$, by art. 202. There are other theorems relating to the tangents which are of use in particular enquiries, of which some were given in *Book I. chap. 8.*

858. When the first fluxion of the ordinate vanishes, if at the same time its second fluxion is positive, the ordinate is then a *minimum*, but is a *maximum* if its second fluxion is then negative; that is, it is less in the former, and greater in the latter case than the ordinates from the adjoining parts of that branch of the curve on either side. This follows from what was shewn at great length in *Chap. 9. B. I.* or may appear thus. Let the ordinate $AF = E$, $AP = x$, and the base being supposed to flow uniformly, the ordinate $PM =$ (art. 751.) E

FIG. 319. $+ \frac{\dot{E}x}{x} + \frac{\ddot{E}x^2}{2x^2} + \frac{\ddot{E}x^3}{6x^3} + \&c.$ let Ap be taken on the other side of A equal to AP, then the ordinate $pm = E - \frac{\dot{E}x}{x} + \frac{\ddot{E}x^2}{2x^2} - \frac{\ddot{E}x^3}{6x^3} + \&c.$ Suppose now $\dot{E} = 0$, then $PM = E + \frac{\ddot{E}x^2}{2x^2}$ &c. and $pm = E - \frac{\ddot{E}x^2}{2x^2} - \&c.$ Therefore if the distances AP and Ap be small enough, PM and pm will both exceed the ordinate AF when $\ddot{E}$ is positive; but will be both less than AF

AF if $\ddot{E}$ be negative. But if $\ddot{E}$ vanish as well as $\dot{E}$, and $\ddot{E}$ does not vanish, one of the adjoining ordinates PM or pm shall be greater than AF, and the other less than it; so that in this case the ordinate is neither a *maximum* nor *minimum*. We always suppose the expression of the ordinate to be positive.

859. In general, if the first fluxion of the ordinate, with its fluxions of several subsequent orders, vanish, the ordinate is a *minimum* or *maximum*, when the number of all those fluxions that vanish is 1, 3, 5, or any odd number. The ordinate is a *minimum*, when the fluxion next to those that vanish is positive; but a *maximum* when this fluxion is negative. This appears from art. 261. or by comparing the values of PM and pm in the last article. But if the number of all the fluxions of the ordinate of the first and subsequent successive orders that vanish be an even number, the ordinate is then neither a *maximum* nor *minimum*.

860. When the fluxion of the ordinate y is supposed equal to nothing, and an equation is thence derived for determining x , if the roots of this equation are all unequal, each gives a value of x that may correspond to a greatest or least ordinate. But if two, or any even number of these roots be equal, the ordinate that corresponds to them is neither a *maximum* nor *minimum*. If an odd number of these roots be equal, there is one *maximum* or *minimum* that corresponds to these roots, and one only. Thus if

$\frac{y}{x} = x^4 + ax^3 + bx^2 + cx + d$, then supposing all the roots of the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ to be real, if the four roots are equal there is no ordinate that is a *maximum* or *minimum*; if two or three of the roots only are equal, there are two ordinates that are *maxima* or *minima*; and if all the roots are unequal there are four such ordinates.

861. To give a few examples of the most simple cases. Let $y = a^2x - x^3$, then $\dot{y} = a^2 - 3x^2$ and $\ddot{y} = -6x$. Suppose $\dot{y} = 0$, and $3x^2 = a^2$ or $x = \frac{a}{\sqrt{3}}$, in which case $\ddot{y} =$

$-\frac{6ax^2}{\sqrt{3}}$. Therefore $\dot{y}$ being negative, y is a *maximum* when

$x = \frac{a}{\sqrt{3}}$, and its greatest value is $\frac{2a^3}{3\sqrt{3}}$. If $y = aa + 2bx - xx$, then $\dot{y} = 2b\dot{x} - 2x\dot{x}$, and $\ddot{y} = -2\dot{x}^2$; consequently y is a *maximum* when $2b - 2x = 0$, or $x = b$. If $y = aa - 2bx + xx$ then $\dot{y} = -2b\dot{x} + 2x\dot{x}$, and $\ddot{y} = 2\dot{x}^2$; consequently y is now a *minimum* when $x = b$, if a be greater than b .

FIG. 321.

862. The right lines BF and GH being perpendicular to the given right line BG in the same plane, H a given point, C any point upon BF, and the figure being supposed to revolve about the axis BG, let it be required to determine the position of the right line HC when the conical surface described by it is a *minimum*. Let DE bisect BG perpendicularly in D and meet HC in E, and (by art. 216.) the surface described by HC about the axis BG will be as $DE \times EH = (\text{supposing } GH = a, DG = b, DE = x, \text{ and consequently } EH^2 = bb + a^2 - x^2)$ $x\sqrt{bb + aa - 2ax + xx}$. Suppose therefore $y = b^2x^2 + a^2x^2 - 2ax^3 + x^4$, then $\dot{y} = 4x^3\dot{x} - 6ax^2\dot{x} + 2a^2x\dot{x} + 2b^2x\dot{x}$, and $\ddot{y} = 12x^2\dot{x}^2 - 12ax\dot{x}^2 + 2a^2\dot{x}^2 + 2b^2\dot{x}^2$. By supposing $\dot{y} = 0$, we have $4xx - 6ax + 2aa + 2bb \times x = 0$; the resolution of which equation gives (besides $x = 0$) $x = \frac{3a + \sqrt{aa - 8bb}}{4}$ or $x = \frac{3a - \sqrt{aa - 8bb}}{4}$. If $GH = \sqrt{2} \times BG$, then $aa = 8bb$, and these two values of x become equal to each other and to $\frac{3a}{4}$. In this case $\frac{\ddot{y}}{x^2} = 12 \times \frac{3a}{4} \times \frac{-a}{4}$.

$+ 2aa + \frac{aa}{4} = 0$, and y is neither a *maximum* nor *minimum*; but while we suppose the point C to move from B along the right line BF, the conical surface described by the right line HC about the axis BG continually increases, though its fluxion vanishes when $DE = \frac{3GH}{4}$. If GH be greater than $\sqrt{2} \times BG$ then $aa > 8bb$, the former value of x gives $\ddot{y}$ positive and y a *minimum*;

imum; but the latter value of x gives $\ddot{y}$ negative, and y a maximum; that is, the value of y is greater when $x = \frac{3a - \sqrt{aa - 8bb}}{4}$ than its adjoining values on either side. But this is not to be understood as if the value of y was then the greatest possible; for it is obvious, that by supposing the point C to proceed in the right line BF, $DE \times EH$ may exceed any given rectangle. See art. 239. When GH is less than $\sqrt{2} \times BG$, aa is less than $8bb$, and the values of x are imaginary. Examples of this kind may frequently occur; and what has been shewn of ordinates, is transferred to the rays that are drawn from a given point to a curve by art. 277.

863. When $\dot{y} = 0$, if $\dot{y}$ be at the same time infinite in respect of x (which is supposed constant,) we cannot conclude that y is then a maximum or minimum without some further enquiry; for the ordinate may then pass through a point of contrary flexure or a cuspid. Let $\frac{\dot{y}}{x} = \frac{\sqrt{ax - xx}}{a}$, then $\frac{\dot{y}}{x^2} =$

$\frac{a - 2x}{2a\sqrt{ax - xx}}$. The supposition of $\dot{y} = 0$ gives $ax - xx = 0$,

and $x = a$, or $x = 0$; in both cases $\ddot{y}$ is infinite; and it is obvious that the curve is reflected from the ordinate, because when x is supposed greater than a , or negative, the values of $\dot{y}$ are imaginary. In like manner if $\dot{y}$, $\ddot{y}$ and $\dddot{y}$ vanish, and $\dot{y}$ be infinite in respect of x , we cannot thence conclude y to be a maximum or minimum. But it may be admitted as a rule, that when $\dot{y} = 0$, and x being constant, $\ddot{y}$ is real and finite; or when any odd number of fluxions of y of the successive orders $\dot{y}$, $\ddot{y}$, $\dddot{y}$, &c. vanish together, and the fluxion of the next order to these is real and finite in respect of x ; we may safely conclude (without any further enquiry) that y is then a maximum or minimum, according as this last fluxion is negative or positive.

However, when after supposing $\dot{y} = 0$, x is determined by a simple

T t t t

simple equation, we may conclude y to be a *maximum* or *minimum* without farther trouble.

864. It was observed in art. 244. that when any quantity N is expressed by a fraction $\frac{P}{Q}$, if P and Q vanish at the same time, we are not thence to conclude that $N = 0$. Thus, suppose $N = \frac{aa - ax}{a - \sqrt{ax}}$; and when $x = a$, the numerator and denominator of N vanish together; but if we reduce the value of N to a more simple form, by dividing the numerator and denominator by their common divisor $\sqrt{a} - \sqrt{x}$, we shall find $N = a \times \frac{\sqrt{a} + \sqrt{x}}{a} = (\text{when } x = a) a \times \frac{2\sqrt{a}}{\sqrt{a}} = 2a$. In such cases the value of N is found by computing $\frac{\dot{P}}{\dot{Q}}$; because when P and Q decrease till they vanish, the ultimate ratio of P to Q is that of $\dot{P}$ to $\dot{Q}$. If P and Q vanish at the same time, then $N = \frac{\dot{P}}{\dot{Q}}$. This rule was given in the

anal. des infiniment petits, p. 145. and is sometimes of use in preventing mistakes concerning the greatest and least ordinates, (such as are described *mem de l'acad. des sciences* 1706.) as well as on other occasions. The computations in enquiries of this kind are sometimes abridged by art. 730. Thus if $mxx = nyy + \frac{m}{m-1} \times yx$, then $y + mx \times x - ny = 0$, and $y + \frac{m}{m-1} \times x - ny = 0$.

865. The greatest and least ordinates are likewise discovered in some cases, by supposing y to be infinite in respect of x ; but it is obvious that there are several exceptions to this rule, since the curve may then form a continued arch that is reflected from the ordinate after touching it, or may be continued on the other side with a contrary flexure. See art. 262. By comparing the signs of y on the different sides of the ordinate (which in this case is a tangent to the curve) the latter of these cases may be distinguished from that wherein the ordinate is a *maximum* or *minimum*; and when the curve is reflected from the

the ordinate, some of the values of y become imaginary on one side of that ordinate. As for the *maxima* and *minima*, which were said to be of the second kind in art. 240. see art. 276.

866. The points of contrary flexure and reflexion are usually determined by supposing $\dot{y} = 0$ or infinite. But this rule being liable to several exceptions, it was shewn in art. 263, that the ordinate y passes through a point of contrary flexure, when, the curve being continued on both sides of the ordinate, $\dot{y}$ is a *maximum* or *minimum*; which, (by what hath been shewn,) doth not always happen when $\dot{y} = 0$ or infinite. Hence

if $\dot{y} = 0$, and $\dot{y}$ be real and finite, then y passes through a point of contrary flexure. This appears likewise by comparing the values of PM and pm in art. 859. Let PM meet the tangent at

FIG. 319.

F in V, and pm meet it in v ; then $PV = E + \frac{\dot{E}x}{x}$, and $pv =$

$E - \frac{\dot{E}x}{x}$. But when $\ddot{E} = 0$, $PM = E + \frac{\dot{E}x}{x} * + \frac{\ddot{E}x^3}{x^3} + \&c.$

and $pm = E - \frac{\dot{E}x}{x} * - \frac{\ddot{E}x^3}{x^3} + \&c.$ consequently if $\ddot{E}$ be po-

sitive, and the distances AP and Ap small enough, PM will be greater than PV and pm less than pv ; and whether $\ddot{E}$ be positive or negative, the arcs FM and Fm shall be on different sides of the tangent Tft; consequently F will be a point of contrary flexure. But if $\ddot{E}$ likewise vanish, and $\ddot{E}$ be of a real value, PM and pm will be both greater or both less than the respective perpendiculars PV and pv intercepted by the tangent; and there will be no point of contrary flexure at F. In gene-

ral, if $\dot{y}, \ddot{y}, \ddot{\dot{y}}, \&c.$ vanish, the number of these fluxions being odd, and the fluxion of the next order to them have a real and finite value, then y passes through a point of contrary flexure; but if the number of these fluxions that vanish be even, it cannot be said to pass through such a point; unless it should be allowed that a double infinitely small flexure can be formed

T t t t 2

at

at one point. To give one of the most simple examples, suppose $y = 1 - x^4$, then $\dot{y} = -4x^3$, $\ddot{y} = -12x^2$, $\dddot{y} = -24x$, and $\ddot{y} = -24x^4$. If we suppose $\ddot{y} = 0$, then $x = 0$, but because $\dot{y}$ is then likewise nothing, and y real and finite, y does not pass through a point of contrary flexure, but is indeed a *maximum*; the truth of which might easily be shewn otherwise.

867. The curve being supposed to be continued from the ordinate PM, or y , on both sides, if $\dot{y}$ be infinite, M is not therefore always a point of contrary flexure, as y is not in this case always a *maximum* or *minimum* by art. 865, and the curve may have its concavity turned the same way on both sides of M. But these cases may be likewise distinguished by comparing the signs of $\dot{y}$ on the different Sides of PM; for when these signs are different M is a point of contrary flexure; for example,

let $y = 1 - x^{\frac{5}{3}}$, then $\dot{y} = -\frac{5}{3}x^{\frac{2}{3}}$, which becomes infinite

when $x = 0$ or $y = 1$, and is affected with contrary signs on different sides of y ; consequently the ordinate passes through a point of contrary flexure when $x = 0$. The suppositions of $\dot{y} = 0$ or infinity, and of $\ddot{y} = 0$ or infinity, serve to direct us where we are to search for the *maxima* or *minima*, and for points of contrary flexure; but where we are not always sure to find them; for though an ordinate or a fluxion that is positive never becomes negative at once, but by decreasing or increasing gradually, (as was shewn in art. 262.) yet after it has decreased till it vanish, it may thereafter increase, continuing still positive; or after increasing till it becomes infinite, it may thereafter decrease, without changing its sign.

868. The points of reflexion, or cuspids, were distinguished into two kinds in art. 268. When the curve is reflected from the ordinate PM or y , it always forms a cuspid; unless when $\dot{y}$ is infinite in respect of x , in which case likewise M is sometimes a cuspid of the second kind; and when $\dot{y}$ or $\ddot{y}$ is real and finite, M is always a cuspid of the second kind. If $\dot{y} = 0$
the

the cuspid may be of either kind. But the most simple kind of the cuspids of the first sort (such as are in some of the lines of the third order) are formed when $\dot{y}$ is infinite, as the most simple kind of points of contrary flexure are formed where $\dot{y} = 0$; see art. 270. and 379. When y is such a *maximum* or *minimum* as was described in art. 865. y passes through a cuspid of the first kind. Other observations may be derived from art. 269.

869. Suppose (as in art. 857.) ST perpendicular from the given point S on the tangent PT in T, SP = r , the fluxion of the curve equal to $\dot{s}$, the fluxion of the ark fP described from the center S equal to $\dot{z}$; consequently $ST = \frac{rz}{\dot{s}}$; and (by art. 281.) P is a point of contrary flexure, when the angle SPT is oblique and ST is a *maximum* or *minimum*; whence rules may be deduced analogous to the former, for determining those points. Suppose z constant, and the fluxion of ST being equal to $\frac{z \dot{r} \dot{s} - r \ddot{s} z}{\dot{s}^2}$; the points of contrary flexure are found by supposing $\dot{r} \dot{s} - r \ddot{s}$, or (because $\dot{s} \dot{s} = \dot{r} \dot{r} + \dot{z} \dot{z}$ and $\dot{s} \ddot{s} = \dot{r} \ddot{r}$) $\dot{s}^2 - r \ddot{r}$, equal to nothing or infinity; but with exceptions similar to those described in art. 866. and 867.

870. Let C be the center of the curvature at M, Cb perpendicular to PM in b, AP = x , PM = y , the ark FM = s , and (by art. 382.) supposing x constant, $Mb = \frac{\dot{s}^2}{\ddot{y}} = \frac{\dot{x}^2 + \dot{y}^2}{\ddot{y}}$ or (because $\dot{s} \dot{s} = \dot{y} \dot{y}$) $Mb = \frac{\dot{s} \dot{y}}{\ddot{y}}$; and the ray of curvature $CM = \frac{\dot{s}^3}{x \ddot{y}}$. For example, if $ay = xx$, then $a\dot{y} = 2x\dot{x}$, $a\ddot{y} = 2\dot{x}^2$, $\dot{x}^2 + \dot{y}^2 = \frac{4xx + aa}{aa} \times \dot{x}^2 = \frac{4y + a}{a} \times \dot{x}^2$, and $Mb =$

FIG. 319

$= \frac{\dot{s}^2}{y} = 2y + \frac{1}{2}a$. If the ray of curvature be expressed by R , the variation of curvature (according to Sir ISAAC NEWTON's explanation) will be as $\frac{\dot{R}}{s}$. But we have insisted on this subject at length in *chap. XI. B. I.*

FIG. 322. 871. Resuming the suppositions in art. 869, let $ST = p$, then the ray of curvature at P , viz. $PC = \frac{rr}{p}$; and if CI be perpendicular to SP in I , $IP = \frac{pr}{p}$. This was demonstrated in art. 384. and may be briefly shewn thus. Let St be perpendicular to pt the tangent at p , and the arcs tn , pu described from the center S meet ST and SP in n and u . Then the angles PCp , TSt being equal, PC will be to ST in the ultimate ratio of Pp to tn ; but IP is to PC in the ultimate ratio of pu to Pp ; consequently IP is to ST in the ultimate ratio of pu to tn , or (because the angles SPp , $S'Tt$ are ultimately equal) of Pu to Tn , that is of r to $\dot{p}$; therefore $IP = \frac{pr}{p}$ and $PC = IP \times \frac{r}{p} = \frac{rr}{p}$. And by substituting for $\dot{p}$ the fluxion of $\frac{rz}{a}$, or (supposing the circle AD to be described with the given radius SA from the center S , SP to meet this circle always in D , $SA = a$, $AD = c$, and consequently $\dot{z} = \frac{rc}{a}$) of $\frac{rrc}{as}$, and supposing $\dot{c}$, $\dot{z}$, $\dot{s}$ or $\dot{r}$ constant, various forms may be derived for expressing the ray of curvature CP , or IP half the chord of the circle of curvature that passes through S . To give one of the most simple examples, let $\dot{s} : \dot{z} :: a^n : r^n$, as in the figures constructed in art. 393. then $p = \frac{rz}{\dot{s}} = \frac{r^{n+1}}{a^n}$, $\frac{\dot{p}}{p} = \frac{n+1}{r} \times \frac{\dot{r}}{r}$, $IP = \frac{pr}{p} = \frac{r}{n+1}$, and $PC = \frac{1}{n+1} \times \frac{an}{r^{n-1}}$.

872. The

872. The rest remaining as in the last article, suppose S to be a radiating point, SP any ray incident upon the curve AP and reflected by it so as to touch *the caustic* at *m*. Then the angle $CPm = CPS$; and the reflected ray Pm will be to the incident ray SP (or r) as 1 is to $\frac{2r}{p} \times \frac{p}{r} \mp 1$, where this unit is to be added or subtracted according as the ark at P has its convexity or concavity towards the radiating point S. For if CR be perpendicular to Pm in R, PR bisected in q , and Ps be taken on the reflected ray equal to the incident ray PS; then (art. 410) $qf : qR :: qR : qm$, and Pq being equal to $\frac{pr}{2p}$, it

follows that $Pm : SP :: \frac{pr}{2p} : r \mp \frac{pr}{2p}$. For example, if $s : z ::$

$a^n : r^n$, then $\frac{r}{p} \times \frac{p}{r} = n \mp 1$, and $Pm : SP :: 1 : 2n \mp 1$.

873. Suppose the curve AP to refract the ray SP, let PM be the refracted ray, and touch *the caustic* in this case at M. The rest of the construction remaining the same as before, let Cr be perpendicular to PM in r , $PR = e$, $Pr = f$, $PM = x$, and let the constant ratio of the sine of incidence to the sine of refraction (or of CR to Cr) be that of n to 1; then (by art. 413.) $PM : rM :: x : x - f$
 $:: CR \times SP \times Pr : Cr \times SR \times PR :: nfr : r \mp e \times e$; con-

sequently $x = \frac{nffr}{nfr - er \pm ee}$, e being equal to $\frac{pr}{p}$ and $f = \frac{r}{np}$
 $\times \sqrt{nn - 1 \times rr \mp pp}$.

874. Suppose the curve AP to be described by any centripetal forces, and the force that acts at any point P will be directly as the square of the velocity at P and inversely as half the chord of the circle of curvature that is in the direction of the force: When it is directed towards a given center S, the area described by the ray SP about S flows uniformly; the velocity at any point P is inversely as ST the perpendicular from S on the tangent, and is to the velocity by which a circle could be described about S at the same distance SP by the same centripetal

FIG. 322.
N. 2.

FIG. 322.
N. 1.

centripetal force as $\sqrt{\frac{\dot{r}}{r}}$ to $\sqrt{\frac{\dot{p}}{p}}$; and the force at P is as $\frac{\dot{p}}{p^2 r}$, because the velocity is as $\frac{1}{p}$, and $PI = \frac{\dot{p}r}{p}$. The same force is as $\frac{\dot{s}\dot{s}}{-r}$ or as $\frac{\dot{z}^2}{r} \pm \ddot{r}$, the fluxion of the area (or rz) being supposed constant. Thus if $\dot{s} : \dot{z} :: a^n : r^n$, the centripetal force directed towards S will be inversely as the power of SP of the exponent $2n + 3$ (because $p = \frac{r^{n+1}}{a^n}$ and $\frac{\dot{p}}{p^2} = \frac{n+1}{r}$ $\times \frac{a^{2n} \dot{r}}{r^{2n+3}}$) and the velocity at P to the velocity in a circle at the same distance as $\sqrt{\frac{\dot{r}}{r}}$ to $\sqrt{\frac{\dot{p}}{p}}$ that is as 1 to $\sqrt{n+1}$. The demonstration that was promised in art. 451, may be deduced in the following manner.

FIG. 323. 875. Let AMB be any figure that can be described by a centripetal force directed towards S that is always as the power of the distance SM of the exponent m . Constitute the angle $ASL : ASM :: m + 3 : 2$; and, supposing $SA = 1$, $SM = x$, $SL = r$, let $r = x^{\frac{m+3}{2}}$; that is, let the angle ASL be to the angle ASM, and the logarithm of the ray SL to the logarithm of SM, always in the same invariable ratio of $m+3$ to 2; then the curve ALD may be described by a centripetal force directed towards S that always varies as the power of the distance SL whose exponent is $\frac{4}{m+3} - 3$. For let SQ and SP be perpendicular to the respective tangents of AM and AL in Q and P, $SQ = y$, and $SP = p$. Then, by the supposition $\frac{\dot{y}}{y^2 x} = ex^m$, where e represents an invariable quantity. By finding the fluents $\frac{1}{y^2} = 2K - \frac{2ex^{m+1}}{m+1}$, where K denotes an invariable

variable quantity according to art. 735. The triangles SMQ,

SLP being similar (art. 394.) it follows that $\frac{I}{p^2} = \frac{x^2}{r^2 y^2} =$

(because $r^2 = x^{m+3}$) $\frac{I}{y^2 x^{m+1}} = \frac{2K}{x^{m+1}} - \frac{2e}{m+1} =$

$2Kr^{\frac{2m+2}{m+3}} - \frac{2e}{m+1}$, and $\frac{\dot{p}}{p^3 r} = \frac{4m+4}{m+3} \times Kr^{\frac{-3m-5}{m+3}}$,

or as the power of r of the exponent $\frac{4}{m+3} - 3$. If the rays from S

be perpendicular to the curve AMB in A and B, and to the curve ALD in A and D, the angle ASD : ASB :: $m+3$: 2, by the construction.

876. Suppose the centripetal force to be always the same at equal distances from the center S. Let e and V denote the forces at the respective distances SA and SP, b and u the velocities at A and P, let SA = a , and SP = r , then $uu = F$. — $2V\dot{r}$ (by art. 435.) in determining which fluent, care must be taken, that u become equal to b when $r = a$. When V is to e as r to a , or as aa to rr , the trajectory is a conic section by art. 445, and 446. And when $V : e :: a^3 : r^3$, the trajectory may be constructed by the areas of conic sections; as has been already shown by several authors. When $V : e :: a^5 : r^5$, the trajectory is constructed in some particular cases only by the areas of conic sections (or circular arcs and logarithms,) but is constructed in general by the arcs of conic sections. In this case a body may continually descend in a spiral line towards the center, and yet never descend so far as to enter within a circle of a certain radius: And a body may recede forever from the center, so as never to arise to a certain finite altitude but revolve in a spiral that is always within a certain circle. This remarkable circumstance could not take place in the trajectories that are described in the former cases, which have been already constructed by others; and therefore we have chosen the construction of this case for an example of the method of determining the trajectory from the law of the centripetal force.

U u u u

877. Let

877. Let b denote the velocity, and AG or GA be the direction of the body at any given point A . Let b be to the velocity with which the body would describe a circle at the same distance by the same centripetal force as $\sqrt{1+mm}$ to $\sqrt{2}$; that is, let $bb : ae :: 1 + mm : 2$. Let SG be perpendicular to AG in G , and any ray SP from S meet the trajectory in P and the circle AX described from the center S in X , $SA = a$, $SG = b$, $SP = r$, ST (the perpendicular on the tangent at P) = p , the ark $AX = c$; and the same fluxions be represented by $\dot{s}$ and $\dot{z}$, as before. Then $uu = F$,

FIG. 324

$$\frac{-2a'er}{r^3} = \frac{a'e}{2r^4} + K = (\text{because when } r = a, \text{ then } uu = bb)$$

$$= \frac{ae}{2} + K = \frac{1+mm}{2} \times ae, \text{ so that } K = \frac{mmae}{2} \frac{a^4 + mmr^4}{2r^4}$$

$$\times ae = bb \times \frac{bb}{pp} = \frac{1+mm}{2} \times \frac{aebbs^2}{rrz^2}; \text{ consequently } \dot{s}^2 : \dot{z}^2$$

$$:: a^4 + mmr^4 : \frac{1+mm}{2} \times bbr^2, \text{ and } \dot{r}^2 : \dot{z}^2 \left(= \frac{rrc^2}{aa} \right) ::$$

$$a^4 + mmr^4 : \frac{1+mm}{2} \times bbr^4 : \frac{1+mm}{2} \times bbr^2. \text{ Therefore}$$

$$\dot{c} = \frac{\dot{r} ab \sqrt{1+mm}}{\sqrt{a^4 - \frac{1+mm}{2} \times bbr^2 + mmr^4}}. \text{ The ratio of } \sqrt{1+mm} \text{ to}$$

1 is that of the velocity at A in the trajectory to the velocity that would be acquired by an infinite descent to A . If $m = 0$,

$$\dot{c} = \frac{\dot{r} ab}{\sqrt{a^4 - bbr^2}}, \text{ and the trajectory is an ark of a circle that passes through } S, \text{ described upon a diameter equal to } \frac{aa}{b}; \text{ which is}$$

agreeable to art. 437.

878. The trajectory is constructed by circular arks and logarithms (and is of that kind of spiral lines which were mentioned at the latter end of art. 343.) when the body sets out from A in the trajectory with a velocity that is to the velocity in a

circle at the same distance SA as SA is to $\sqrt{SA^2 + \sqrt{SA^2 - SG^4}}$.

In this case (supposing $SA^2 : SG^2 :: n : 1$, or $aa = nbb$) $1 + mm :$

$$2 :: a^2 : a^2 + \sqrt{a^4 - b^4} :: n : n + \sqrt{nn - 1}, m = \frac{n}{n + \sqrt{nn - 1}}$$

$\pm \sqrt{nn-1}$, and $1 + mm \times bb = \frac{2aa}{n + \sqrt{nn-1}} = 2maa$; conse-

quently $\dot{c} = \frac{\mp raa\sqrt{2m}}{aa-mrr}$. Suppose 1. that the velocity at A

in the trajectory is to the velocity in a circle at the distance

SA as $\sqrt{n}$ to $\sqrt{n + \sqrt{nn-1}}$, (in which case $m = n - \sqrt{nn-1}$)

upon SA produced take $Sk : SA :: 1 : \sqrt{m}$, describe the

circle kxK from the center S, take the ark kK (on the same side

of k that AG is of A) equal to $\frac{1}{\sqrt{2}} \times \log. \frac{1-\sqrt{m}}{1+\sqrt{m}}$, the modulus

being Sk , join SK and it shall be the tangent of the trajectory

at the point S. To find any other point of the trajectory as

P; let $SK = d$, take the ark $Kx = \frac{1}{\sqrt{2}} \times \log. \frac{d-r}{d+r}$, join Sx

and upon the right line Sx take $SP = r$. For suppose the ark

$Kx = y$, then by art. 731. $\dot{y} = \frac{\mp ddr\sqrt{2}}{dd-rr} =$ (because $dd :$

$aa :: 1 : m$) $\frac{\mp aar\sqrt{2}}{aa-mrr}$ and $\dot{c} = \frac{\dot{y}}{d} = \frac{\mp aar\sqrt{2m}}{aa-mrr}$ as it ought

to be. Therefore describe an equilateral hyperbola Knv

having its center in S and vertex in K; let any right line

Srn meet the hyperbola in n and the tangent at K in r ,

then let the circular sector $SKx : SKn :: \sqrt{2} : 1$, and SP be

taken upon Sx equal to Kr , and P shall be a point in this tra-

jectory. 2. Let the velocity at A in the trajectory be to the

velocity in a circle at the distance SA as $\sqrt{n}$ to $\sqrt{n - \sqrt{nn-1}}$, FIG. 325.

then $m = n + \sqrt{nn-1}$, SK is to be taken less than SA in the

ratio of 1 to $\sqrt{m}$, the sector $SKx : SKn :: \sqrt{2} : 1$; and SP

is to be taken upon the ray Sx , so that $Kr : SK :: SK : SP$.

879. In the first case (when the velocity at A in the tra- FIG. 324.

jectory is to the velocity in a circle at the same distance as a to

$\sqrt{a^2 + \sqrt{a^2 - b^2}}$) if the body set out from A with the directi-

on GA, it will perform its revolutions in a spiral always with-

in the circle Kxz , and never can arise to the altitude SK from

the center S; because Kr (to which the distance SP is always

equal,

equal,) cannot become equal to SK, while the area SK n or ark K x are finite. The area described by the ray SP about the center S is always to the hyperbolic area generated by the right line rn in the invariable ratio of $\sqrt{2}$ to 1; because $\dot{z} : \dot{y}$

$$:: r\sqrt{m} : a, \frac{r\dot{z}}{2} = \frac{yrr\sqrt{m}}{2a} = \frac{+ arrr\sqrt{2m}}{2aa-2mrr}; \text{ and the fluxi-}$$

on of the area K rn ($= SKn - SKr$) is $\frac{arrr\sqrt{m}}{2aa-2mrr}$. Therefore

if the body set out from A, with the direction AG, it will descend in the curve APS to the center S in the time that by proceeding in the tangent AG with its velocity at A, it would describe about S a triangle equal to $\sqrt{2} \times KRN$, KR being supposed equal to SA. In this figure the area SoP x K (terminated by the curve SoP the circular ark K x and right lines SK and P x) admits of a perfect quadrature, and is to the triangle SK r as $\sqrt{2}$ to 1.

FIG. 325.

880. In the second case, when the velocity at A is to the velocity in a circle at the same distance as a to $\sqrt{a^2 - \sqrt{a^4 - b^4}}$, if the body set out from A with the direction AG, it will revolve in a spiral that always approaches to the circle K x , but it never can descend to this circle; because SP ($= \frac{SK^2}{Kr}$) cannot become equal to SK in any finite time. This spiral has an asymptote at a distance from S equal to $\frac{SA^2}{SK} \times \sqrt{2}$, because,

by art. 877. $pp = \frac{1 + mm}{a^4 + mmr^4} \times bbr^4$, and the ultimate value

of pp is $\frac{1 + mm}{mm} \times bb = (\text{in this case}) \frac{aa\sqrt{2}}{d}$.

FIG. 326.

881. In other cases, the trajectory may be constructed by hyperbolic and elliptic arks, from art. 805. If the velocity at A be to the velocity in a circle at the distance SA as $\sqrt{1 - mm}$ to $\sqrt{2}$, and the direction at A be perpendicular to SA (or $a = b$) then by substituting in art. 877. $-mm$ for mm ,

$c = 1$.

$$\dot{c} = \frac{-\dot{r}aa\sqrt{1-mm}}{\sqrt{a^4-1-mm} \times aarr-mmrr} = \frac{-\dot{r}aa\sqrt{1-mm}}{\sqrt{aa-rr} \times \sqrt{aa+mrr}}, \text{ which}$$

may be compared with $\frac{-\dot{b}bp}{\sqrt{aa-pp} \times \sqrt{bb+pp}}$, by supposing $b =$

$\frac{a}{m}$ and $p = r$. The fluent of this last fluxion was found (art.

805. fig. 308.) to be equal to $\frac{b}{a} \times AR + AE - EP$. Therefore, when the velocity at the distance SA is less than the velocity by which a circle would be described at the same distance in the ratio of $\sqrt{1-mm}$ to $\sqrt{2}$, the trajectory may be constructed in the following manner. Let $SD : SA :: 1 : m$, $Sb : SA :: \sqrt{1+mm} : 1$, describe an hyperbola AEZ having SA and SD for its two semi-axes, and an ellipse ARb having SA and Sb for its semi-axes; draw Ep a tangent to the hyperbola at any point E and Sp a perpendicular to Ep; upon SA take $SQ = Sp$, and let the ordinate at Q meet the ellipse in R; then upon the circle Axz described from the centre S take the ark $Ax : \frac{1}{m} \times AR + AE - EP :: m\sqrt{1-mm} : 1$, upon the ray Sx take $SP = Sp$, then P shall be a point in the trajectory. In this case the velocity at A is such as could be acquired by a body descending to A from some greater distance by the same centripetal force.

882. When the velocity at the distance SA is to the velocity in a circle at the same distance as $\sqrt{1+mm}$ to $\sqrt{2}$, then

$$\dot{c} = \frac{+\dot{r}aa\sqrt{1+mm}}{\sqrt{aa-rr} \times \sqrt{aa-mmrr}} = (\text{by supposing } pp = aa - rr)$$

$$+ \frac{\dot{p}aa\sqrt{1+mm}}{\sqrt{aa-pp} \times \sqrt{1-mm} \times aa + mmpp}; \text{ and by comparing this fluxion}$$

with that in art. 805. it appears that when m is less than 1, we are to take $SD : SA :: \sqrt{1-mm} : m$, $Sb : SA :: 1 : \sqrt{1-mm}$, and to proceed in the construction as in the last article; only after Sp and Ax are determined, we are now to take SP upon the ray Sx equal to $\sqrt{SA^2 - Sp^2}$.

883. When m is greater than 1, then by supposing $p = \frac{a\sqrt{rr-aa}}{r}$

and consequently $r = \frac{aa}{\sqrt{aa-pp}}$, $\dot{c} = \frac{\dot{p}aa\sqrt{1+mm}}{\sqrt{aa-pp} \times \sqrt{mm-1} \times aa + pp}$

therefore in this case we are to make $SD : SA :: \sqrt{mm-1} : 1$, $Sb : SA :: m : \sqrt{mm-1}$, to determine Sp and Ax , as in art. 881, and then we are to take SP (upon the ray Sx) equal to a third proportional to $\sqrt{SA^2 - Sp^2}$ and SA . If upon Sx you take SP a third proportional to Sp and SA , P will be a point in the trajectory which is described by a centrifugal force directed from S that is inversely as the fifth power of the distance. When the direction of the body at the distance SA is oblique to the ray drawn from the center S , the trajectories may be constructed in a similar manner.

FIG. 319. 884. If the curve FM be described by powers directed in any manner whatsoever, and the force at any point M , resulting from the composition of these powers, act in the direction MK , and be measured by MK ; let MK be resolved into the force MO in the direction of the ordinates $MP (= y)$ and the force OK parallel to the base $AP (= x)$ then, the time being supposed to flow uniformly, or the velocity at M being represented by the fluxion of the curve FM , the force MO will be measured by $\dot{y}$, and the force OK by $\dot{x}$, by art. 465 and 466. but we insisted on this, and its use, in *Book I. chap. XI. article 465, &c.*

FIG. 327. 885. Let a body descend along the curve FPA by its gravitation towards S , the time of the motion be represented by t , the velocity at any distance SP or r by u , the centripetal force at the same distance by g , the ark FP by s ; then the motion of the body along the curve is accelerated by the force $\frac{-gr}{s} =$
 $\frac{\dot{u}}{\dot{t}} = \left(\text{because } \dot{t} = \frac{\dot{s}}{u} \right) \frac{u\dot{u}}{\dot{s}}$; consequently $u\dot{u} = -gr$, $u\dot{u}$
 $= F. - 2gr$ and $\dot{t} = \frac{\dot{s}}{\sqrt{F. - 2gr}}$. When the gravity is uniform

niform and acts in parallel lines, let z be the space described in a vertical line from the beginning of the descent, then u

$$= F. 2gz = 2gz, \dot{t} = \frac{z}{\sqrt{2gz}}, \text{ and } t = \sqrt{\frac{2z}{g}}. \text{ The gra-}$$

vity being still uniform, let the body begin to descend along the curve DMS from D, MN be perpendicular to the horizon-
tal line DA in N, the ark SM = s , MN = z , and t repre-
sent the time of descent from M to the lowermost point S, FIG. 328.
N. 1.

then $\dot{t} = \frac{s}{\sqrt{2gz}}$. If DMS be an ark of a semi-cycloid that has its axis perpendicular to the horizon, the diameter of the generating circle = a , AS = b , then (by the second property of this fi-

$$\text{gure in art. 805.) } s : -z :: \sqrt{a} : \sqrt{b-z}, \text{ and } \dot{t} = \frac{-z\sqrt{a}}{\sqrt{2gz \times b-z}}.$$

If N be to 1 as the semi-circumference of a circle to the dia-
meter, N shall represent the fluent of $\frac{-z}{2\sqrt{z \times b-z}}$ that is gene-

rated while z becomes equal to b ; consequently the time of
descent in the ark of the cycloid DMS is expressed by $N \times$
 $\sqrt{\frac{2a}{g}}$, and is to the time of descent in the axis a (viz. $\sqrt{\frac{2a}{g}}$)
as N to 1; as we found in art. 408.

886. But when DMS is an ark of a circle t is a fluent of a
higher kind, and is not to be represented by the areas of conic
sections, but by their arks. Let C be the center of the circle,
HCS the vertical diameter, MV perpendicular to HS in V, FIG. 328.
N. 2.
HS = E , CA = F ; then $s : -z :: CS : MV :: \frac{1}{2}E :$

$$\sqrt{\frac{1}{4}EE - FF - 2Fz - zz} \text{ and } \dot{t} = \frac{-Ez}{2\sqrt{gz} \times \sqrt{\frac{1}{4}EE - FF - 2Fz - zz}}.$$

Let this fluxion be compared with $(\dot{Q} =) \frac{-bbz}{2\sqrt{az} \times \sqrt{bb - 2az - zz}}$
the fluent of which was determined in art. 805. and we have

$$bb = \frac{1}{4}EE - FF, \text{ or } b = AD, 2F = 2e = \frac{bb - aa}{a} \text{ and } a = \frac{1}{2}E.$$

$\frac{1}{2} E - F = SA$. Therefore let S be the center, A the vertex, and SD the asymptote of the hyperbola AE ; produce HD till it meet Sb perpendicular to SA in k , take $Sb = Dk$, and describe the ellipsis ARb ; let SQ or $SP = \sqrt{ax}$; and the fluent Q will be represented by $\frac{AD}{SA} \times AR + AE - EP$, by art. 805. t the time of descent from M to S will be expressed by $Q \times \frac{HS \sqrt{SA}}{AD^2 \sqrt{2g}}$, and is to the time of descent in the vertical SA as Q to $\frac{AD^2}{CS}$.

887. It follows from art. 807, that if the semi-circumference be to the diameter as N to 1, and $HA : AD : m :: 1$, then the time in the whole ark DMS will be represented by $\frac{HS \times N}{\sqrt{2g \times HA}} \times$

$1 - \frac{1}{4mm} + \frac{9}{64m^4} - \&c.$ the ultimate value of which, when SA is supposed to vanish, is $\sqrt{\frac{HS}{2g}} \times N$. Therefore the time of descent in the ark DMS is to this ultimate value of t (which is said to be the time in an evanescent ark, and by art. 885. is equal to the time in any ark of a cycloid that has the diameter of the generating circle equal to CS) as $\frac{\sqrt{mm+1}}{m} \times$

$1 - \frac{1}{4mm} + \frac{9}{64m^4} - \frac{9}{256m^6} + \&c.$ to 1. By the sequel of the same art. 807. if $SH : SA :: n : 1$, then the whole time in the ark DMS will be expressed by $N \sqrt{\frac{E}{2g}} \times$

$1 + \frac{1}{4n} + \frac{9}{64n^2} + \&c.$ and the time in DMS will be to the time in an infinitely small ark (or the ultimate value of t) as

$1 + \frac{1}{4n} + \frac{9}{64n^2} + \frac{25}{256n^3} + \&c.$ to 1. When DMS is

is a quadrant, the time of descent is measured by the arks of the *lemniscata*, of which we gave an easy construction in article 803. FIG. 307.

888. If a body descend or ascend in the vertical line z in a *medium*, and the resistance be represented by R , its motion is accelerated or retarded by $g \pm R = \frac{\dot{u}}{t} = \frac{\dot{uu}}{z}$; and $\dot{u} =$

$\overline{g \pm R} \times \dot{z}$. For example, if the resistance be as the square of the velocity, and a denote the velocity when the resistance is equal to the gravity, or $R : g :: uu : aa$, then $\dot{u} = g \dot{z}$

$\times \frac{aa \pm uu}{aa}$, $\dot{z} = \frac{aa}{g} \times \frac{\dot{u}}{aa \pm uu}$, and $\dot{t} = \frac{\dot{u}}{g \pm R} = \frac{aa}{g} \times$

$\frac{\dot{u}}{aa \pm uu}$; whence z and t may be computed from u by logarithms or circular arks. See art. 542. When the body descends along a curve line, it is accelerated by the excess of the force $\frac{g}{s}$ above R which is therefore equal to $\frac{uu}{s}$; and if it ascends

the sum of these forces is equal to $-\frac{uu}{s}$. When a trajectory

is described in a *medium*, and the centripetal force is directed towards S , let this force at any point P be to the centripetal force at P by which the same trajectory would be described in a void as z to a , and (retaining the same symbols as in art. 869.) FIG. 322.

the resistance at P will be as $\frac{z}{pps}$, or, if the area of the figure be

supposed to flow uniformly, as $\dot{z} \dot{s}$, (by art. 452.) and is to the centripetal force at P in the *medium* as $pr \dot{z}$ to $2z \dot{s} p$. If the resistance R be in the compound ratio of the density D and square of the velocity uu , then D is as $\frac{R}{uu}$ or (because uu is as

$\frac{az}{pp}$) as $\frac{\dot{z}}{azs}$: And if the curve be such as can be described in

X x x x

a

a void by a force directed towards S that is as any power of the distance, D will be inversely as $\frac{r\dot{s}}{r}$. If the centripetal force in the medium be uniform and act in parallel lines, and y be an ordinate in the direction of the force, then the resistance will be to the gravity as $y\ddot{s}$ to $2y\dot{s}$; and if R be as Duu , then D will be as $\frac{y\ddot{s}}{y\dot{s}}$.

FIG. 327. 889. Suppose FPA to be the figure which is assumed by a chain that is perfectly flexible, and gravitates towards the given point S . Then ST the perpendicular from S on PT the tangent at P shall be inversely as $F.gr$, and the tension of the chain at any point P inversely as ST , by art. 567. If FPA be the line of swiftest descent from F to the lowermost point A , $SA = a$, $SP (= r)$ meet the circle AD described from the center S in D , the ark $AD = c$, and u be to a as the velocity at P to the velocity acquired at A ; then $c = \frac{aur}{r\sqrt{rr-uu}}$ by art. 581. and 582. If the gravity act in parallel lines, let $PM (= y)$ be an ordinate in the direction of the force, $FM = x$, $PM = y$, the ark $FP = s$, then if FPA be the *catenaria*, $\frac{x}{s}$ will be as $F.gy$ by article 568. And if FPA be the line of swiftest descent, u denote the velocity acquired at P (or $u = \sqrt{F.2gy}$) and a the velocity acquired at the lowermost point A , then $s : x :: a : u$ by art. 575. and 576.

FIG. 319. 890. The base AP being represented by x , and the ordinate PM by y , if the $F.yx$, be computed, and the expression be made to vanish when $x = 0$, according to art. 735. it will give the area $APMF$. When the fluent is negative, it gives the area on the other side of PM . For example, let $y = x^m$, then $F.yx = F.x^m x = \frac{x^{m+1}}{m+1}$ which gives the area when m is any positive number, or is a negative number less than 1. But when m is

m is a negative number greater than unit, this expression is negative, and gives the area on the other side of PM. The area generated by the ray SP about S (according to the symbols in

FIG. 322.

art. 869.) is the fluent of $\frac{r^2}{2}$ or of $\frac{rrc}{2a}$. We have had many examples above of the computation of areas from those theorems. There are several general theorems, for computing the area, described above, as in art. 752, 819, 830, 832, &c.

891. The solid generated by the area APMF about the axis

FIG. 319.

AP is found by computing $F. 2Ny^2 x$, where N denotes the ratio of the semi-circumference to the diameter. For example, let the figure be any conic section, AP the axis, and the general equation of the figure being $yy = Axx + Bx + C$, the solid

generated by APMF about AP will be equal to $\frac{2NAx^3}{3}$

+ $NBx^2 + 2NCx$. Let Ap be taken on the other side of A equal to AP and pm be the ordinate at p, then $pm^2 = Axx - Bx + C$; consequently the solid, generated by the area ApmF

about the axis Ap, will be equal to $\frac{2NAx^3}{3} - NBx^2 + 2NCx$.

Therefore the solid generated by the area PMmp is equal to $\frac{4NAx^3}{3} + 4NCx$. When $x = 0$, $yy = C$, consequently the

cylinder generated by the rectangle PHbp (HFb being parallel to Pp) is equal to $4NCx$; and the excess of the frustum generated by the area PMmp above this cylinder is $\frac{4N}{3} \times Ax^3 =$

(supposing $Pp = 2x = v$) $\frac{NAv^3}{6}$; which (if $PZ : Pp :: \sqrt{A} : 1$) is $\frac{1}{4}$ of the cone generated by the right-angled triangle PZp about Pp, and is always of the same magnitude when v and A are the same. The frustum is greater or less than the cylinder, according as A is positive or negative; and they are equal when $A = 0$, that is when the figure is a Parabola. In this manner the properties of these solids described above p. 24. are briefly demonstrated. When the value of $F. 2Nyyx$

X x x x 2

is

is negative, it represents the solid that is generated by the area on the other side of the ordinate PM. Thus if $y = x^{-m}$, then

$$F. 2N_{yyyx} = \frac{2Nx^{-2m+1}}{-2m+1} = \frac{-2N_{yyyx}}{2m-1}, \text{ which expression is}$$

negative when m is greater than $\frac{1}{2}$, and represents the limit to which the solid generated by the hyperbolic area on the other side of PM continually approaches while that area is supposed to be produced. See art. 307. &c.

892. The ark FP is the fluent of $\dot{s}$ or of $\sqrt{\dot{x}^2 + \dot{y}^2}$. For

$$\text{example, let } ayy = x^3, \text{ then } y = \frac{x^{\frac{3}{2}}}{a^{\frac{1}{2}}}, \dot{y} = \frac{3x^{\frac{1}{2}}\dot{x}}{2a^{\frac{1}{2}}}, \dot{s} = \frac{\dot{x}}{2} \times$$

$$\sqrt{\frac{9x+4a}{a}}, \text{ and by art. 727. } s = \frac{9x+4a|^{\frac{3}{2}}}{27a^{\frac{1}{2}}} + K. \text{ If we}$$

$$\text{suppose } s = 0 \text{ when } x = 0, \text{ then } K = \frac{-8a}{27}, \text{ and } s =$$

$$\frac{9x+4a|^{\frac{3}{2}} - 8a^{\frac{3}{2}}}{27a^{\frac{1}{2}}}. \text{ In like manner, if we make use of the no-}$$

FIG. 322. N. I. tation in art. 869. $\dot{s} = \frac{rr}{\sqrt{rr-pp}}$. Suppose, for example, app

$$= r^3, \text{ then } \dot{s} = \frac{\dot{rr} \sqrt{a}}{\sqrt{arr-rrr}} = \dot{r} a^{\frac{1}{2}} \times a-r|^{-\frac{1}{2}}, \text{ and (art. 727.)}$$

FIG. 329. $s = 2a^{\frac{1}{2}} \times a-r|^{-\frac{1}{2}} = 2\sqrt{aa-ar}$. If we suppose AP to be a parabola, S the focus and A the vertex, then T will be always found in the right line AE perpendicular to SA; and the parabolic ark AP = PT + log. ST + TA, the modulus being SA. For let SA = a, ST = p, SP = r, AP = s, and PT = q,

$$\text{then } pp = ar, \dot{s} = \frac{\dot{rr}}{\sqrt{rr-pp}} = \frac{\dot{rr}}{\sqrt{rr-ar}} = (\text{because } q =$$

$$\sqrt{rr-ar} \text{ and } \dot{q} = \frac{2rr-ar}{2\sqrt{rr-ar}}) \dot{q} + \frac{\dot{ar}}{2\sqrt{rr-ar}}. \text{ But if } u = ST + TA$$

+ TA = $\sqrt{ra} + \sqrt{ra-aa}$, then $\frac{\dot{u}}{u} = \frac{\dot{r}}{2\sqrt{rr-ar}}$; consequently

$\dot{s} = \dot{q} + \frac{a\dot{u}}{u}$, and $s = q + \log. u$ the modulus being equal to a .

See art. 746. and 845. for the mensuration of circular arks, and art. 806. 807. 808. for hyperbolic and elliptic arks.

893. The surface generated by the ark s , when the figure revolves about the base, (the ordinate being represented by y and base by x), is F. $4Nys$ or F. $4Ny\sqrt{x^2+y^2}$, by art. 229. Thus if the parabola AP revolve about the axis ASM, PM being perpendicular to AS in M, PM = $y = 2AT = 2\sqrt{ar-aa}$, and

$\dot{s} = \dot{r}\sqrt{\frac{r}{r-a}}$; consequently $y\dot{s} = 2\dot{r}\sqrt{ar}$, the surface generated

by the ark AP is $\frac{16N}{3} \times r\sqrt{ra} + K = \frac{16N}{3} \times \overline{SP \times ST - SA^2}$

and (if SE be a mean proportional betwixt SP and ST) this surface is to the circle of the radius AE as 8 to 3.

894. Let C be the center, CD half the transverse axis, and CA half the second axis of the ellipse ADB, F the focus, PN perpendicular to CD, PM perpendicular to CA, and PK perpendicular to the curve meet CD in K, CA = a , CD = b , CF = c , CN = x , PN = y , and the ark AP = s ; then NK : NC :: $a^2 : b^2$, or NK = $\frac{a^2x}{b^2}$, PN² = $\frac{aa}{bb} \times$

$\overline{bb - xx}$, PK² = $a^2 - \frac{a^2c^2x^2}{b^4}$ and PK = $\frac{a}{bb} \times \sqrt{b^4 - c^2x^2}$.

But $\dot{s} : \dot{x} :: PK : PN = y$, $y\dot{s} = \frac{ax\sqrt{b^2 - c^2x^2}}{bb} =$ (sup-

posing $c : b :: b : d = CG$) $\frac{acx\sqrt{dd-xx}}{bb}$: Therefore let CA

and NP meet the circle GZE described from the center C in E and Z, and when the figure is supposed to revolve about the axis CD, the surface generated by the elliptic ark AP will be to the area CEZN as $4N \times ac$ to bb ; and if DI perpendicular to CD meet GZE in I, the whole surface of the spheroid will

will be to the surface of the sphere of the radius CA as $\frac{4Nac}{bb} \times$
 CEID to $4Naa$, that is as $EI + CA$ to $2CA$. In like manner
 if PK produced meet AC in k , $Mk : MC (=y) :: b^2 : a^2$, and
 $Pk = \frac{b}{aa} \times \sqrt{a^4 + c^2 y^2}$; let $DP = f$, and $\dot{f} : \dot{y} :: Pk : PM$, or
 $PM \times \dot{f} = \dot{y} \times Pk$; consequently the fluxion of the surface ge-
 nerated by the ark DP about the axis CA is $\frac{4Nby}{aa} \times \sqrt{a^4 + c^2 y^2}$
 $=$ (if $c : a :: a : e = Cg$) $\frac{4Nbc}{aa} \times y \sqrt{ee + yy}$ the fluent of
 which is $\frac{2Nb}{e} \times y \sqrt{ee + yy} + 2Nb \times \log. y + \sqrt{ee + yy}$, (the
modulus being equal to e or Cg) $= 2N \times CM \times Pk + 2Nb \times$
 $\log. CM + Pk \times \frac{Cg}{CD}$. Hence the surface generated by the
 elliptic quadrant DPA about the axis CA is $2Nb \times$
 $b + \log. a \times \frac{b + c}{c}$; and the surface of this spheroid is to the
 surface of a sphere of the radius CD as $CD + \log. \frac{Cg \times CA}{DF}$
 to $2CD$, the *modulus* being Cg . These constructions agree with
 Mr. COTES's *Harmon. mensurar.* p. 28. and 29. where he illu-
 strates the transition from circular arks to logarithms (or
 from the measures of angles to the measures of ratios,) that so
 often occurs in the resolution of the various cases of a problem,
 from an analogous transition observed long ago by VIETA in
 the resolution of cubic equations; the roots of which are in
 some cases obtained by trisecting an ark, and in other cases by
 what may be called the trisecting a ratio (*i. e.* interposing two
 mean proportionals betwixt the terms of the ratio;) so that the
 trigonometrical and logarithmical canon are mutually supple-
 ments to each other. The harmony of those measures, which was
 so much considered by this excellent Author, may be further
 illustrated by the resolution of the two following useful pro-
 blems relating to the spheroid.

895. In plain sailing the meridians are supposed parallel, and the degrees of longitude as well as those of latitude are supposed equal; whereas the meridians intersect each other in the pole, the degrees of longitude decrease in the same proportion as the semi-diameters of the parallels of latitude, and the degrees of latitude (because of the oblate figure of the earth) increase from the equator towards the Poles. In order to correct some of the errors that arise in Navigation from these false suppositions, a projection was invented (commonly called *Mer-
cator's chart*) in which the meridians are still supposed parallel, and the degrees of longitude enlarged as in the former, but the degrees of latitude upon the meridians are enlarged in the same proportion. The arcs of the meridian thus enlarged, (or the *me-
ridional parts*) are found in a sphere or spheroid by the following theorems. Let the ark DH, or angle DCH, be the latitude for which the meridional parts z are required, HE its sine, let CT bisect the ark Hd (the complement of HD) and meet the tan-

gent at d in T. Then 1. in the sphere $z = \log. \frac{CD}{dT}$ the *modu-* FIG. 331.
N. 1.

lus being CD. 2. In the oblate spheroid, let Db be an ark whose sine eb is to EH as CF the distance of the focus from the center to CD the semi-diameter of the equator, let Ct bisect the ark db and meet dT in t ; then $z = \log. \frac{CD}{dT} - \frac{CF}{CD} \times$

$\log. \frac{CD}{dt}$. 3. In the oblong spheroid let Dq be the ark whose FIG. 331.
N. 2. tangent is to EH the sine of DH as CF to CD, and $z = \log. \frac{CD}{dT} + \frac{CF}{CD} \times Dq$.

896. For, supposing ADB to be a meridian section through the poles A and B as in art. 894. let $CA = a$, $CD = b$, $CF = c$, $CM = y$, $EH = u$, and the elliptic ark $DP = s$. Then to find the meridional parts z , we are to suppose the element or fluxion of the ark DP to be always enlarged in the ratio of CD the radius of the equator to PM the radius of the parallel

of P, that is $\dot{z} = \dot{s} \times \frac{CD}{PM} = (\text{because } \dot{s} : \dot{y} :: PK : NK ::$

CH

CH : CE) $\frac{by}{\sqrt{bb-uu}} \times \frac{CD}{PM} =$ (because PM:NK :: bb:aa, & NK:

CE :: PN : EH) $\frac{y}{y} \times \frac{aau}{bb-uu}$. By what we found in art.

894. $Pk = \frac{b}{aa} \times \sqrt{a^4 + ccyy}$ or $\frac{b}{aa} \times \sqrt{a^4 - ccyy}$, according

as CD is greater or less than CA; consequently Pk be-

ing to Mk as CE to EH, we have $u = \frac{bby}{\sqrt{a^4 + ccyy}}$ or $y =$

$\frac{aau}{\sqrt{b^4 + ccuu}}$, and (by art. 728.) $\frac{y}{y} = \frac{u}{u} + \frac{ccuu}{b^4 + ccuu} = \frac{u}{u}$

$\times \frac{b^4}{b^4 + ccuu}$. Therefore $\dot{z} = \frac{aab^4\dot{u}}{bb-uu \times b^4 + ccuu}$, that is $\dot{z} =$

$\frac{bb\dot{u}}{bb-uu}$ in the sphere, $\dot{z} = \frac{bb\dot{u}}{bb-uu} - \frac{b^2cc\dot{u}}{b^4 - ccuu}$ in the ob-

late spheroid, and $\dot{z} = \frac{bb\dot{u}}{bb-uu} + \frac{b^2cc\dot{u}}{b^4 + ccuu}$ in the oblong

spheroid. Suppose now db to be the diameter of the circle dDb ,

join dH and Hb , then the triangles TdC and dHb will be si-

milar, $dT : dC :: dH : Hb :: \sqrt{Cd-EH} : \sqrt{Cd+EH}$, $dT =$

$b \times \sqrt{\frac{b-u}{b+u}}$, and the modulus being b the fluxion of $\log. \frac{CD}{dT}$

(or of $\log. \sqrt{\frac{b+u}{b-u}}$) shall be $\frac{bb\dot{u}}{bb-uu}$. In like manner, because

$eb = \frac{cu}{b}$, $dt = b \times \sqrt{\frac{bb-uu}{bb+cu}}$, the fluxion of $\log. \frac{CD}{dt}$ (or

of $\log. \sqrt{\frac{bb+cu}{bb-uu}}$) is $\frac{b^2cc\dot{u}}{b^4 - ccuu}$. Therefore in the sphere $\dot{z}$

$= \log. \frac{CD}{Td} = \log. CD - \log. dT$: And in the oblate spher-

oid, $\dot{z} = \log. \frac{CD}{dT} - \frac{CF}{CD} \times \log. \frac{CD}{dt}$. In the oblong spheroid,

roid Dq is the fluent of $\frac{b^3 cu}{b^3 + ccuu}$; consequently $z = \log.$

$$\frac{CD}{aT} + \frac{CF}{CD} \times Dq.$$

897. These logarithms are hyperbolic, or of NAPIER's first sort; but it is easy to adapt the theorems to the tabular logarithms, and to express the meridional Parts in minutes, as is usual. Thus in the sphere subtract the logarithmic tangent of half the complement of the latitude from the logarithm of the radius (or 10.000000) and multiply the remainder by 7915.704467897 &c. (*viz.* the number of minutes contained in the radius divided by the *modulus* of the tables,) then the product shall give the meridional parts in minutes.

898. In the oblate spheroid we have this easy rule, let $CF : CD :: 1 : n$, and u be the sine of the given latitude DCH for which the meridional parts are required. The table for the meridional parts being already computed in the sphere, find the meridional parts in this table for the latitude whose sine is $\frac{1}{n} \times u$, divide these by n , subtract the quotient from the meridional parts in the same table for the given latitude DCH ; and the remainder shall be the meridional parts for the same latitude in the oblate spheroid. This problem is resolved by infinite series, and a table of the meridional parts is computed for the oblate spheroid wherein $cc : bb :: 22 : 1000$, in an ingenious treatise published lately by the Reverend Mr. MURDOCH, whose table may be examined by this rule; and it may likewise serve for facilitating the computation, when a different ratio is assumed for that of cc to bb . The greatest difference betwixt the meridional parts in the oblate spheroid and sphere is easily computed by finding the meridional parts in the sphere for the latitude whose sine is to the radius as 1 to n , and dividing these by n .

899. In the oblong spheroid, to find the meridional parts for the latitude whose sine is u , add to the meridional parts in the sphere for the same latitude $\frac{1}{n} \times Dq$, Dq being the ark whose tangent is $\frac{1}{n} \times u$.

Y y y y

900. Let

FIG. 332. 900. Let $PMoux$ be a pyramid of an uniform density upon the rectangular base $Moux$, and suppose that the attraction of its particles is inversely as the power of the distance of any exponent n less than 3. Let the attraction at the given distance $PC (=a)$ be represented by e , the attraction at any distance $PM (=r)$ by V ; then if the angles MP_0, MP_x be continually diminished the gravity at P towards the pyramid $PMoux$

will be ultimately as $\frac{Vr}{3-n} \times Mo \times Mx$. For if $Nnkl$ be a section

of the pyramid parallel to $Moux$ at a distance $PN = PC$, the gravity at P towards the pyramid shall be ultimately equal to the fluent of $Vr \times Mo \times Mx$ or (because $Nl : Mx :: Nn : Mo$

$:: a : r$ and $V : e :: a^n : r^n$) of $\frac{a^{n-2}er}{r^{n-2}} \times Nl \times Nn$, that is to

$$\frac{a^n er^{3-n}}{3-n} \times Nl \times Nn = \frac{Vr}{3-n} \times Mo \times Mx.$$

901. Hence it will appear (by proceeding as in art. 642.) that if a portion of a solid contained by planes that intersect each other in PH attract a particle at P , PMA be one of these planes, the right line PM meet the circle BNC described from the center P with the given distance PC in N , MQ be perpendicular to PC and NR to PH , $PC = a$, $PM = r$, $PQ = z$, $PR = x$, the sine of the inclination of the planes to the radius as f to a ; and supposing this angle to be diminished continually till it vanish, the ultimate value of the gravity at P towards the slice of the solid contained by the planes in the direction PC be represented by q , then q will be equal to the

fluent of $\frac{fVr^2zx}{3-n \times aa}$ or of $\frac{a^{n-2}efzx}{3-n} \times r^{2-n}$. If PC coincide

with PH , then $x : a :: z : r$, and the gravity at P will be as the

fluent of $\frac{a^{n-3}ef}{3-n} \times r^{3-n}xx$. If PH be perpendicular to PC ,

the gravity at P towards the portion of the solid will be ultimately

mately as the fluent of $\frac{a^{n-3}ef}{3-n} \times r^{3-n} \sqrt{aa-xx}$, because in this case $r : z :: a : \sqrt{aa-xx}$.

902. Suppose a solid to be generated by the figure PMA revolving about the axis PC, and the gravity at P towards this solid in the direction PC to be measured by Q, then Q =

$$\frac{4Na^{n-2}e}{3-n} \times F. r^{3-n} \sqrt{aa-xx}, N \text{ being supposed to denote the ratio}$$

of the semi-circumference to the diameter, as formerly. For example, if PMA be a semicircle of the radius PC, then PM =

$$2PR \text{ or } r = 2x; \text{ and } Q = F. \frac{Nea^{n-2}r^{4-n}}{3-n} = \left(\text{when } r \text{ be-} \right.$$

$$\text{comes equal to } 2a) \frac{2^{5-n} \times Na^3e}{3-n \times 5-n}. \text{ If } n=2, Q = \frac{8Na^3e}{3}, \text{ and the}$$

gravity is the same as if the whole matter in the sphere attracted from the center C, because the solid content of the sphere

is $\frac{8Na^3}{3}$; and in other cases the gravity is to this attraction as

$3 \times 2^{2-n}$ to $3-n \times 5-n$. If $n = -1$ these are equal. If $n = 0$, their ratio is that of 4 to 5; and if $n = 1$, the ratio is that of 3 to 4. In different spheres, when n is given, the gravity is as PC^{3-n} because e is as PC^{-n} .

903. To find the attraction at the pole A towards the spheroid generated by the semi-ellipsis ADB about the axis AB, Fig. 333.

suppose P to coincide with A, AC = a, CD = b, CF (F being the focus) = c, AR = x, AQ = z, AM = r; then

$$AR^2 :: NR^2 :: AQ^2 : QM^2, \text{ or } xx : aa - xx :: zz : \frac{bb}{aa} \times 2az - zz :: z : \frac{bb}{aa} \times 2a - z; \text{ consequently } z = \frac{2b^2ax^2}{a^4 + c^2x^2},$$

$$\text{and (because } x : a :: z : r) r = \frac{2b^2a^2x}{a^4 + c^2x^2} \text{ or } x = \frac{aa}{ar} \times$$

$$\sqrt{bb \pm \sqrt{b^4 - c^2rr}}. \text{ Therefore } Q = \frac{2^{5-n}Na^{4-n}b^{6-2n}e}{3-n} \times$$

Y y y y 2

F.

$\times F. \frac{x^{4-n} \dot{x}}{a^4 + c^2 x^2}^{3-n}$; or, if we suppose $bb = dc$, Q will be e-

qual to $\frac{4Na^n + 2b^2c}{3-n \times c^4} \times F. \frac{2d^2cr + cr^2\dot{r}}{r^n \sqrt{dd - rr}} - F. \frac{2b^2\dot{r}}{r^n}$, which

fluents are easily measured by the areas of conic sections, when n is any integer number. The upper signs are for the oblong spheroid.

FIG. 334. 904. To find the attraction at the point D in the equator of the spheroid, let P coincide with D , DBE be a section of the solid perpendicular to its equator, PH or DH a tangent at D , HNe a circle described from the center D with the radius Dc ($= CA$) meet DM in N , MQ perpendicular to DE and NR to DH , $CA = a$, $CD = b$, $CF = c$ as formerly, and $DQ = z$, $DM = r$, $DR = x$, Then $NR^2 : DR^2 :: DQ^2 : QM^2$, that

is $aa - xx : xx :: zz : \frac{aa}{bb} \times 2bz - zz :: z : \frac{aa}{bb} \times 2b - z$ and z

$= 2baa \times \frac{aa - xx}{a^4 + c^2 x^2}$; consequently $r (= \frac{az}{\sqrt{aa - xx}}) =$

$\frac{2ba \sqrt{aa - xx}}{a^4 + c^2 x^2}$. Therefore $q = \frac{a^{n-3}ef}{3-n} \times F. r^{3-n} \dot{x} \sqrt{aa - xx}$

$= F. \frac{8a^6 b^3 fex}{3-n \times 2aab^{1n}} \times \frac{aa - xx}{a^4 + c^2 x^2}^{\frac{4-n}{2}}$, which gives the ultimate value of the gravity at D towards a slice of the spheroid

contained by two planes perpendicular to its equator that intersect each other in DH , when the angle contained by the planes vanishes, by art. 901. If we suppose $c = 0$ or $a = b$, the last fluxion becomes equal to $\frac{8a^{n-3}efx}{3-n \times 2^n} \times \frac{aa - xx}{aa - xx}^{\frac{4-n}{2}} =$ (sup-

posing $yy = aa - xx$) $\frac{a^{n-3}ef}{3-n \times 2^n} \times \frac{-y^{5-n}\dot{y}}{\sqrt{aa - yy}}$, the fluent of

which gives the ultimate value of the gravity at D towards the slice of the sphere (described upon the diameter of the equator of

of the spheroid) that is contained by the same planes. Because the sections of the spheroid by planes perpendicular to the equator are ellipses similar to the meridian section and to one another, and the sections of the sphere by these planes are circles, the gravity at D towards the spheroid is to the gravity at D towards the sphere described upon the diameter of the equator as the former to the latter fluent, that is (supposing cc :

$$aa :: m : 1) \text{ as } F. a^3 - n b^3 - n \dot{x} \times \frac{aa - xx^{\frac{4-n}{2}}}{aa + mxx^{\frac{3-n}{2}}} \text{ to } F. \dot{x} \times$$

$aa - xx^{\frac{4-n}{2}}$. These fluxionary expressions are rational when n is an even number; and when n is an odd number, they are transformed into rational expressions by supposing $x =$

$\frac{az}{\sqrt{aa + zz}}$. Hence, therefore, the gravity at the equator, as well as the gravity at the poles, is measured by circular arcs or logarithms when n is any integer number less than $+3$.

905. When $n = 2$, the gravity at the pole or equator is easily computed from the first theorem in art. 901. viz. $q = F.$

$$\frac{a^{n-2}efzx}{3-n} \times r^{2-n} = (\text{when } n = 2) F. efzx. \text{ For when the par-}$$

ticle P, whose gravity is required, is at A, as in art. 903. z (or

$$AQ, \text{ supposing } AR = x,) = \frac{2bb}{a} \times \frac{xx}{aa + mxx} \text{ and } q = \frac{2b^2ef}{a}$$

$$\times F. \frac{x^2 \dot{x}}{aa + mxx}; \text{ consequently the gravity at the pole A to-}$$

wards the spheroid is to the gravity at A towards the sphere of

$$\text{the diameter AB as } bb \times F. \frac{x^2 \dot{x}}{aa + mxx} \text{ to } \frac{1}{3} aa. \text{ When the}$$

particle P is at D on the circumference of the equator, suppose,

$$\text{as in art. 904. } DR = x, \text{ then } DQ = z = 2b \times \frac{aa - xx}{aa + mxx},$$

$$\text{and } q = F. efzx = 2bef \times F. \dot{x} \times \frac{aa - xx}{aa + mxx}; \text{ consequently the}$$

gravity at D on the circumference of the equator towards the
sphe-

spheroid is to the gravity at D towards the sphere upon the diameter DE as $F. abx \times \frac{aa - xx}{aa + mxx}$ to $F. x \times aa - xx =$ (when $x = a$) $\frac{2a^3}{3}$. And these fluents give the same constructions by circular arks and logarithms that were described in art. 646. and 647. The gravity at any point P on the surface of the spheroid in the direction parallel to the axis, or perpendicular to it, may be computed in like manner from the theorem $q = F. efzx$; but this case is reduced to the former by art. 634. When the density varies, but so as to be uniform over any surface similar and concentric to ADBE, the gravity at any place in the plane of the equator, or axis of the spheroid, may be computed by art. 668, &c. The reader will find this subject treated in a different manner in a late ingenious essay, *Phil. Trans.* N. 449. by Mr. CLAIRAUT. It was demonstrated in art. 636. &c. that if the density of the earth was uniform, its figure would be such a spheroid as is generated by an ellipsis revolving about its second axis, according to the theory of gravity; but this was assumed as an hypothesis in art. 679. 681, &c. where the density was supposed variable.

FIG. 335. 906. The centers of gravity and oscillation of figures are determined from art. 509. and 534. Let G be the center of gravity, and O be the center of oscillation of the plane figure FfmM when it revolves about the axis Ff, OGA perpendicular to Ff in A and to Mm in P, AP = x, Mm = y, GA = z, OA = u; then $z = \frac{F. yxx}{F. yx}$

and $u = \frac{F. yx^2 \dot{x}}{F. yxx}$. Thus if $\frac{1}{2} y = x^m$, $z = \frac{F. x^{m+1} \dot{x}}{F. x^m \dot{x}} =$

$\frac{m+1}{m+2} \times x$, or GA : PA :: m + 1 : m + 2; and $u = \frac{F. x^{m+2} \dot{x}}{F. x^{m+1} \dot{x}} =$

$\frac{m+2}{m+3} \times x$, or OA : PA :: m + 2 : m + 3. The center of

FIG. 239. oscillation of a sphere was determined in art. 536. from the fluent

fluent of $2ny^2y \times \overline{aa - yy}$ (supposing in fig. 239. the radius FIG. 239.
 $GE = a, GN = PM = y, OG = z$ and n to 1 as the circumfe-
 rence of the circle to its radius) which is $2ny^3 \times \frac{aa - yy}{3}$;
 and this fluent becomes equal to $\frac{4na^5}{15}$ when $PM = GE$ or $y = a$,
 which being divided by $\frac{2n}{3} \times a^3 \times z$ (the solid content of the
 sphere multiplied by the distance of its center of gravity from
 the axis of oscillation) gives $\frac{2}{5} \times \frac{aa}{z} = u$. The center of
 percussion is in a right line perpendicular to AO at O . Several
 principles concerning the center of gravity and its motion, that
 are of use in the resolution of problems were explained in art.
 511. 526. 533. 544. 551. &c. The motion of a fluid issuing
 from a cylindric vessel was considered in art. 537. 540. 541, &c.
 and an example of the method by which the principle con-
 cerning the equality of the ascent and descent of the center of
 gravity is applied to this enquiry (*comment. petropol. Tom. 2.*)
 is described in art. 544. But the same theory has been since
 prosecuted more fully by the learned Author, and illustrated
 by various experiments, in a particular treatise, entituled, *Hy-*
drodynamica.

907. In any engine the proportion of the power to the weight,
 when they balance each other, is found by supposing the en-
 gine to move, and reducing their velocities to the respective
 directions in which they act; for the inverse ratio of those ve-
 locities is that of the power to the weight, according to the
 general principle of mechanics. But it is of use to determine
 likewise the proportion they ought to bear to each other, that
 when the power prevails, and the engine is in motion, it may
 produce the greatest effect in a given time. When the power
 prevails, the weight moves at first with an accelerated motion;
 and when the velocity of the power is invariable, its action up-
 on the weight decreases while the velocity of the weight in-
 creases. Thus the action of a stream of water or air upon a
 wheel is to be estimated from the excess of the velocity of the
 fluid

fluid above the velocity of the part of the engine which it strikes, or their relative velocity, only. The motion of the engine ceases to be accelerated when this relative velocity is so far diminished, that the action of the power becomes equal to the resistance of the engine arising from the gravity of the matter that is elevated by it, and from friction; for when these balance each other, the engine proceeds with the uniform motion it has acquired. Let a denote the velocity of the stream, u the velocity of the part of the engine which it strikes when the motion of the machine is uniform, and $a - u$ will represent their relative velocity. Let A represent the weight which would balance the force of the stream when its velocity is a , and p the weight which would balance the force of the same stream if its velocity

was only $a - u$; then $p : A :: \overline{a - u}^2 : a^2$, or $p = \frac{A \times \overline{a - u}^2}{aa}$

and p shall represent the action of the stream upon the wheel. If we abstract from friction, and have regard to the quantity of the weight only, let it be equal to qA (or be to A as q to 1,) and because the motion of the machine is supposed uniform,

$p = q \times A = \frac{A \times \overline{a - u}^2}{aa}$, or $q = \frac{\overline{a - u}^2}{aa}$. The momentum of

this weight is $qAu = \frac{Au \times \overline{a - u}^2}{aa}$; which is a maximum when

the fluxion of $\frac{u \times \overline{a - u}^2}{aa}$ vanishes, that is when $\dot{u} \times \overline{a - u}^2 -$

$2uu \times \overline{a - u} = 0$, or $a - 3u = 0$. Therefore in this case the machine will have the greatest effect if $u = \frac{a}{3}$, or the weight

$qA = \frac{A \times \overline{a - u}^2}{aa} = \frac{4A}{9}$; that is, if the weight that is raised

by the engine be less than the weight which would balance the power in the proportion of 4 to 9; and the momentum of the

weight is $\frac{4Aa}{27}$.

908. If

908. If we would likewise consider the friction arising from the motion of the weight, let 1 be to n as the weight is to the resistance of the engine which would arise from this friction, if the motion of the engine was such that the part of it impelled by the stream moved with the given velocity a ; then supposing the friction to be always in the compound ratio of the weight and velocity, the resistance of the engine arising from the same cause when the part of the wheel impelled by the stream moves with the velocity u will be $\frac{nqAu}{a}$. Suppose there-

fore $p = qA + \frac{nqAu}{a} = \frac{A \times \overline{a-u}^2}{aa}$, then $qA = \frac{A}{a} \times \frac{\overline{a-u}^2}{a+nu}$,

and the *momentum* of the weight $qAu = \frac{Au}{a} \times \frac{\overline{a-u}^2}{a+nu}$; the fluxion of which being supposed to vanish, we shall find $aa - 3au - 2nuu = 0$ or $u = \frac{2a}{3 + \sqrt{9+8n}}$, and the weight $qA = 4A \times$

$\frac{1 + \sqrt{9+8n}}{3 + \sqrt{9+8n}}$; that is, the machine will have the greatest effect (according to this supposition) when $u : a :: 2 : 3 + \sqrt{9+8n}$, and the weight is to that which would balance the power as $2 + 2\sqrt{9+8n}$ to $9 + 4n + 3\sqrt{9+8n}$. For example, if $n = \frac{7}{8}$, then $u = \frac{2a}{7}$ and $qA : A :: 20 : 49$; consequently, though the velocity u be less than in the former case in the ratio of 6 to 7, (and therefore the action of the power on the wheel be greater,) yet the weight that is raised is less in the ratio of 45 to 49, and the effect of the engine is less in the ratio of 270. to 343. If n be very small in respect of 1, then $u : a :: 1 : 3 + \frac{2n}{3}$, and $qA : A :: 4 + \frac{4n}{3} : 9 + 4n$ nearly. But if we

would have likewise regard to the friction arising from the motion of the parts of the engine, as well as to that which arises from the elevation of the weight, the computation will be somewhat

Z z z z

what different. Let the friction be equal to mA when the machine moves without any charge in such a manner that the velocity of the part impelled by the stream is equal to a ; and the friction will be equal to $\frac{mAu}{a}$ when this velocity is u ; where we suppose m invariable, because the machine remains the same. When the motion of the engine is uniform,

$p = qA + \frac{nqAu}{a} + \frac{mAu}{a} = \frac{A \times a - u^2}{aa}$, and supposing the momentum of qA to be a maximum, u will be found by resolving the equation $u^3 + \frac{3}{2n} - 1 - \frac{m}{2} \times au^2 - \frac{2+m}{n} \times aau +$

$\frac{a^3}{2n}$. For example if $n = \frac{1}{10}$ and $m = \frac{1}{10}$, u is nearly $\frac{3a}{10}$, qA is about $\frac{39A}{103}$, and the effect of the engine about $\frac{1}{3}$ of Aa or $\frac{1}{3}$ ths of what it would have been if there was no friction and u was equal to $\frac{a}{3}$.

FIG. 336. 909. Suppose that the given weight P descending by its gravity in the vertical line raises a given weight W by the line PMW (that passes over the pully M) along the inclined plane BD , the height of which BA is given; and let the position of the plane BD be required, along which W will be raised in the least time from the horizontal line AD to B . Let $AB = a$, $BD = x$, $t =$ time in which W describes DB ; the force which accelerates the motion of W is $P - \frac{aW}{x}$, tt is as $\frac{xx}{Px - aW}$, and if we suppose the fluxion of this quantity to vanish, we shall find $x = \frac{2aW}{P}$ or $P = \frac{2aW}{x}$; consequently the plane BD required is that upon which a weight equal to $2W$ would be sustained by P ; or if BC be the plane upon which W would sustain P , then $BD = 2BC$. But if the position of the plane BD be given, and W being supposed variable, it be required to find the ratio of W to P when the greatest momentum is produced

duced in W along the given plane BD; in this case W ought to be to P as BD to $BA + \sqrt{BD+BA} \times \sqrt{BA}$.

910. The radius CA and angle ACB being given, let E be any point upon the ark AB, EM the sine of the angle ECA, EN the sine of the angle ECB, n any positive number, and let it be required to determine the point E when $EM^n \times EN$ is a *maximum*. Upon AC produced beyond C, take $CD : CA :: n-1 : n+1$; draw DG parallel to CB, meeting the circle AB in G, and if CE bisect the angle ACG, it will meet the circle in the point E required. For, let ER parallel to CB meet CA in R, $CR = x$, $ER = y$, and when $y^n x$ is a *maximum* (or when its fluxion vanishes) $\frac{ny}{y} + \frac{x}{x} = 0$, by art. 728. or $nx =$

$-\frac{yx}{y}$. Let the tangent at E meet CA in T, CB in Z, and CG in Q; and AP perpendicular to CE meet CB in K and the circle again in H; then $RT = \frac{yx}{y} = nx$, or $RT : CR :: n : 1 ::$

$ET : EZ :: AP : PK :: PH : PK$; consequently $HK : KA :: n-1 : n+1 :: CD : CA$, and DH is parallel to CB. Therefore H coincides with G, and the ark GA is bisected in E when $ER^n \times CR$ is a *maximum*, or (because ER is to EM and CR to EN in the same invariable ratio of the radius to the sine of the given angle ACB) when $EM^n \times EN$ is a *maximum*.

911. Let a fluid that moves with the velocity and direction AC strike the plane CE; and suppose that this plane moves parallel to itself in the direction CB. Take $CD : CA :: 1 : 3$, draw DG parallel to CB meeting the circle AB in G; and if the plane CE bisect the angle ACG, then the effect of the fluid upon CE will be greatest, at the beginning of the motion. But if the plane CE has already acquired a motion in the direction CB, let its velocity in this direction be to the velocity of the stream as Aa to AC, and let Aa be parallel to CB; let a circle described from the center C, with the distance Ca meet DG in g; and the effect of the stream upon the plane CE will be greatest

FIG. 337.
N. 2.

Z Z Z Z 2

test

test in this case when the plane bisects the angle aCg . For let AP and ap be perpendicular to CE in P and p , and Ab perpendicular to AP in b ; then the motion of the particles of the fluid in the direction perpendicular to CE will be represented by AP , their motion in the direction parallel to the plane CE by CP , the motion of the plane in the former direction by Ab , and its motion in the latter direction by ab . The action of the fluid on the plane depends on their relative velocity only, that is, on the difference of the motions AP and Ab , (which is equal to $bP = ap$) and on the sum or difference of the motions PC and ba , which is equal to pC . It follows, that the action of the stream on the plane CE is the same in this case as when the plane is at rest and the stream strikes it with the direction and force aC . Let this force aC be resolved into ap perpendicular to the plane CE and pC parallel to it; and because the latter has no effect upon the plane CE , let the force ap be resolved into the force ak parallel to CB and pk perpendicular to it; then because the force pk has no effect to impell the plane CE in the direction CB , ak will measure the force with which any particle of the fluid impells CE in the direction CB ; and the number of particles incident upon the plane CE in the same time being as ap , the effect of the stream to move the plane CE in the direction CB shall be measured by $ak \times ap = (Em \text{ and } EN \text{ being perpendicular to } Ca \text{ and } CB \text{ in } m \text{ and } N, \text{ and consequently } ak : ap :: EN : CE)$

$$\frac{ap \times ap \times EN}{CE} = Em^2 \times EN \times \frac{Ca^2}{CE^3}, \text{ which is a maximum when } CE$$

bisects the angle aCg by the last article; because CE and Ca are supposed to be given, $n = 2$, and $CD : CA :: n - 1 : n + 1 :: 1 : 3$. If $Aa = 0$, g coincides with G , and the stream has the greatest effect when CE bisects the angle ACG .

FIG. 338.

912. Let CV be perpendicular to Aa in V , and CE produced meet Aa in t ; take $VL = VC \times \sqrt{2}$; $Vf = \frac{3Va}{2}$, join Lf , and Vt the tangent of the angle VCE (VC being radius) shall be equal to $Lf + Vf$, when the plane CE is in the most advantageous position, CA the velocity and direction of the stream and Aa the velocity and direction of the plane CE being given. For
let

let pq perpendicular to CV in q meet Ca in u , let aC and VC produced meet Dg in z and d ; then because $aC = 3 Cu$, $at = 3pu =$ (because $ag = 2ap$) $\frac{3gz}{2} = \frac{3gd + 3dz}{2} = \frac{3gd}{2} + \frac{Va}{2}$
 $=$ (because $gd^2 = Cg^2 - Cd^2 = Ca^2 - CV^2$) $\frac{8CV^2}{9} + Va^2 = \frac{4}{9} \times 2CV^2 + \frac{9Va^2}{4} = \frac{4}{9} \times$
 $VL^2 + Vf^2 = \frac{4}{9} \times Lf^2$, and $\frac{3gd}{2} = Lf$) $Lf + \frac{Va}{2}$; and $Vt = at + Va = Lf + Vf$. If $CV = a$, $Va = c$, then $Vt = \sqrt{2aa + \frac{9cc}{4}} + \frac{3c}{2}$. The negative sign is to take place when aCB is greater than a right angle.

913 When the angle ACB is right, A coincides with V , Fig. 338.
 DG is perpendicular to AD , and $At = Lf + Af = \sqrt{2aa + \frac{9cc}{4}}$ N. 2.

$+ \frac{3c}{2}$. If $Aa = c = 0$, then $At : AC :: \sqrt{2} : 1$, or AP the sine of ACE to the radius AC as AG to $2CA$ or as $\sqrt{AD}$ to $\sqrt{2CA}$, that is as $\sqrt{2}$ to $\sqrt{3}$. Therefore the stream at the beginning of the motion will have the greatest effect upon the plane CE , if the angle ACE be of $54^\circ. 44'$. and this is the case which has been considered by several Authors: But if the plane CE has already a motion in the direction CB , the stream will have the greatest effect upon it if the angle ACE be greater. For example, if the velocity of the plane CE in the direction CB be a third part of the velocity of the stream, or $c = \frac{a}{3}$, then

$At = \sqrt{2aa + \frac{aa}{4}} + \frac{a}{2} = 2a$, or the tangent of the angle ACE ought to be double of the radius, that is $ACE = 63^\circ. 26'$. If $c : a :: \sqrt{8} : \sqrt{9}$ then $At : AC :: 2 + \sqrt{2} : 1$, and ACE ought to be of $73^\circ. 40'$. If $c = a$, then $ACE = 74^\circ. 19'$.

914. Hence the sails of a common wind-mill ought to be so situated that the wind may strike them in a greater angle than that

that of $54^{\circ} 44'$; for this is the most advantageous angle at the beginning of the motion only; and when any part of the engine has acquired a velocity c , the effect of the wind upon that part will be greatest, when the tangent of the angle in which the wind strikes it is to the radius as $\sqrt{2 + \frac{9cc}{4aa}} + \frac{3c}{2a}$ to 1.

Let the right line bb represent the length of one of the sails, take AC to Ab as the velocity of the wind to the velocity of the given point b about the axis of motion, $LA = AC \times \sqrt{2}$, and a being any point upon bb , take $Af = \frac{3Aa}{2}$; then if the sail be so formed that the wind shall strike it at any distance Aa from the axis of motion in an angle whose tangent is always to the radius as $Lf + Af$ to CA , the wind shall have the greatest effect upon the sail. It is true, that a celebrated Author has drawn an opposite conclusion from his computations, *viz.* that the angle in which the wind strikes the sail ought to decrease, as the distance from the axis of motion increases; that if $c = a$, the wind ought to strike the sail in an angle of 45° ; and that if the sail be in one plane, it ought to be inclined to the wind at a *medium* in an angle of about 50 degrees: But if he had reduced the equation of six dimensions, by which he has determined the *maximum*, to a biquadratic equation, our conclusions would have agreed; and the divisor by which this reduction may be made, is of no use for determining the most advantageous position of the sail when the engine is in motion; because it does not give a *maximum*, but a *minimum* that corresponds to the case when CE coincides with Ca , and the stream has no effect upon the plane CE . Suppose $Aa = AC$, or $c = a$; and if the angle ACE be of 45° , CE will coincide with Ca , the velocities of the plane CE and of the stream estimated in the direction perpendicular to CE must be equal; so that the stream will have no effect upon the plane CE in this case to preserve or accelerate its motion; and the angle ACE must be increased, that the velocity of the stream in the direction ap (in which it acts upon the plane) may be greater than the velocity of the plane in the same direction. In the same manner, it is

is obvious that if Aa was equal to $2AC$, and ACE of $54^\circ 44'$ then the stream could have no effect upon the plane CE , and the angle ACE must be increased.

915. When the engine is of such a nature that the whole fluid, or the same quantity of it, is always incident on the plane CE in its various positions, the force by which it impells CE in the direction CB is as $ak = Em \times EN \times \frac{Ca}{CE^2}$, which is a *ma-*

FIG. 339.

ximum (Ca and CE being given) when CE bisects the angle aCB by art. 910. because in this case $n=1$, $CD : CV :: n-1 : n+1 :: 0 : 2$, that is CD vanishes and DG coincides with CB . In this case, if AC and Aa , the velocities of the stream and plane, be given, with CB the direction of the motion of the plane, but the angle ACB be variable, and Aa be greater than $\frac{1}{2} AC$, the action of the fluid upon the plane will not be greatest when AC is perpendicular to CE and CE to CB ; but when ACB being an obtuse angle, the sine of ACV is to the radius as AC to $2Aa$ and the plane CE is perpendicular to AC . For let $Cg = Ca$, aq be perpendicular to CB in q , then $ak = \frac{1}{2} gq$. Suppose $CA = a$, $Aa = c$, $AV = x$, then $ak = Cg \mp Cq = Ca + aV = \sqrt{aa+cc-2cx} + x - c$; and when the fluxion of this quantity

vanishes, $\frac{-cx}{\sqrt{aa+cc-2cx}} + \dot{x} = 0$, $aa + cc - 2cx = cc$, $aa = 2cx$,

or $x : a :: a : 2c$; and it is easy to see from the construction that in this case ACE must be a right angle. For example, if $c = a$ then $x = \frac{1}{2} a$, $ACV = 30$ degr. $ACB = 120$ degr. $ACE = 90$ degr. and $ECB = 30$ degr.

916. Suppose now that AC represents the direction and velocity of the wind, CB the direction in which a ship moves, Aa parallel to CB the velocity of the ship, CE the situation of the sail, and let us abstract from her *lee-ward* way, or suppose that no deflexion from the direction CB is occasioned by the obliquity of the wind or sail to the course CB . Then, in order to determine the most advantageous position of the sail CE , (when CA , CB and Aa are given) that the wind may act with the greatest force to impell the ship in the given direction CB , produce AC till

FIG. 338.

till $AD : AC :: 4 : 3$. let DG be parallel to CB , and a circle aeg described from the center C with the distance Ca meet DG in g ; then the sail CE ought to bisect the angle aCg , by art. 911: or let CV be perpendicular to Aa in V , $LV = VC \times \sqrt{2}$, $Vf = \frac{3}{2} Va$, $Vt = Lf + Vf$, and CE produced pass through t . When Aa the velocity of the ship is neglected, or when the motion begins, CE ought to bisect the angle ACG ; which is the case that was resolved long ago by Mr. FATIO and Mr. HUYGENS by a biquadratic equation; and has been considered more fully since by Mr. BERNOULLI, *Manoeuvre des Vaisseaux*, chap. 5. But in some cases the ratio of Aa to AC is not inconsiderable; and supposing AC perpendicular to CB , if (for example) $Aa = \frac{1}{3} AC$, the angle ACE ought to exceed $\frac{1}{2} ACG$ ($= 54^\circ 44'$ in this case) by about $9 \frac{2}{3}$ degr. if we would have the wind impell the ship with the greatest force in the direction CB .

917. The force with which the wind impells the ship in the Direction CB is always measured by $ak \times ap$; and when this force and the resistance of the water become equal, the motion of the ship becomes uniform. Let CK represent the uniform velocity which the ship would acquire by the same wind in its direction AC if the sail was perpendicular to AC , and the force in this case which sustains the motion of the ship, and balances the resistance, will be measured by KB^2 . Therefore (the resistance of the water being as the square of the velocity of the ship) $CK^2 : Aa^2 :: KB^2 : ap \times ak =$ (supposing Aa parallel to CB to meet CE in t) $at^2 \times \frac{EN^2}{CE^3}$; consequently $Aa : at :: CK$

$\times \sqrt{\frac{EN^3}{CE^3}} : KB$. Let $CA = a$, $Aa = x$, $EN = y$, $AV = p$, $CK : KB :: 1 : m$; then $Aa : at :: y\sqrt{y} : ma\sqrt{a}$; and because $At = Vt + AV = \sqrt{aa - pp} \times \frac{\sqrt{aa - yy}}{y} + p$, $Aa = x = \frac{\sqrt{aa - pp} \times \sqrt{aay - yy} + py\sqrt{y}}{ma\sqrt{a} + y\sqrt{y}}$. Suppose a , p and m to be constant

stant, and when x is a *maximum* we shall find that $aa - 3yy - \frac{2y\sqrt{ay}}{m} + \frac{3py\sqrt{aa-yy}}{\sqrt{aa-pp}} = 0$. This is an equation for determining the sine of the angle ECB which ought to be contained by the sail and the line of the ship's motion, in order that the velocity of the ship in this line may be the greatest possible, a , p and m being given.

918. If AC be perpendicular to CB, then $p=0$, and $3yy + \frac{2y\sqrt{ay}}{m} = aa$. For example, let $m = 2\sqrt{2}$, that is, let the velocity of the ship be to the velocity of the wind when the ship moves in the direction of the wind and the wind is perpendicular to the sail as 1 to $1 + 2\sqrt{2}$ (or nearly as 1 to 3.828.) then if the ship sail in a direction perpendicular to that of the wind, the sail ought to be inclined to the wind in an angle of 60° , or to the way of the ship in an angle of 30° . For the equation for y when x is a *maximum* is, in this example, $aa - 3yy - \frac{y\sqrt{ay}}{\sqrt{2}} = 0$, which gives $y = \frac{a}{2}$; and in this case the velocity of the ship is $\frac{a\sqrt{y} \times \sqrt{aa-yy}}{ma\sqrt{a} + y\sqrt{y}} = \frac{a}{3\sqrt{3}}$. The sine of the angle ECB is always less than $\frac{1}{\sqrt{3}} \times CB$.

919. The angle ECB contained by the sail CE and course of the ship CB, with AC the velocity of the wind being given, the velocity of the ship is greatest when ACE is a right angle, that is, when the wind is perpendicular to the sail; as is obvious, and agrees with art. 917. where, if a , y and m be given, x becomes a *maximum* when $p=y$. Supposing AC to be perpendicular to CE, $x = \frac{aa\sqrt{y}}{ma\sqrt{a} + y\sqrt{y}}$ and is a *maximum* when y or $p = a \times \sqrt[3]{\frac{mm}{4}}$; that is, of all cases wherein the wind is supposed to be perpendicular to the sail, the velocity of the ship is greatest

A a a a a

FIG. 340.

rest (providing CK be not less than $\frac{1}{3}$ CA, or m be not greater than 2,) when the sine of the angle ECB contained by the sail and course, is to the radius as $\sqrt[3]{mm}$ to $\sqrt[3]{4}$, and the velocity of the ship is greater in this case than when the wind blows in the direction of the course and is perpendicular to the sail in the ratio of $m + 1$ to $3\sqrt[3]{\frac{mm}{4}}$, or (supposing $n = 2 - m$) of $1 - \frac{n}{3}$ to $1 - \frac{n^2}{2}$. If, for example, CK : CA :: 1 : 2, the velocity of the ship in the direction CB will be greatest when the sine of ECB, or ACV, is to the radius as 1 to $\sqrt[3]{4}$; that is when the angle ECB is about $39^\circ 3'$, or when the angle ACb, in which the direction of the wind is inclined to the course of the ship, is an angle of about $50^\circ 57'$. And the velocity of the ship is in this case greater than when the same wind blows directly in the course of the ship, and the sail is perpendicular to the wind, (in which case the wind is commonly thought to be most favourable,) in the ratio of $\sqrt[3]{32}$ to $\sqrt[3]{27}$ or of $2\sqrt[3]{4}$ to 3; and by inclining the sail CE to the wind, so as to increase the angle BCE, the velocity of the ship in the right line CB will be still greater. There may be many other cases supposed from art. 916. wherein a side-wind would promote the motion of the vessel more than a direct wind. For example, if the velocity of the vessel in the direction CB be to the velocity of the wind as 1 to 3, and the angle ACB be only of $109^\circ 28'$, the force by which the wind will promote the motion of the vessel in the course CB will in this case be greater than when the wind is direct, or the angle ACB is of 180° , in the ratio of $\sqrt[3]{32}$ to $\sqrt[3]{27}$; the sail being supposed in both cases to have the most advantageous position, which was determined in art. 916.

FIG. 341. 920. A given line AC being divided in B, the rectangle AB $\times$ BC is a *maximum* when AB = BC, by what is shown in the elements of Geometry. Hence it follows, that if a given line AG

AG be divided into a given number of parts AB, BC, CD, DE, EF, FG, the product of the parts $AB \times BC \times CD \times DE \times \&c.$ is a *maximum* when they are equal to each other; because if BD the sum of any two adjoining parts be divided equally in C and inequally in c , $BC \times CD$ is greater than $Bc \times cD$, and $AB \times BC \times CD \times DE \times \&c.$ is greater than $AB \times Bc \times cD \times DE \times \&c.$ If a given right line AG be divided in C, and $AC^n \times CG^m$ be a *maximum*, then $AC : CG :: n : m$; for if we suppose $AG = a$, $AC = x$, $x^n \times \overline{a-x}^m = y$, then $\frac{\dot{y}}{y} = \frac{n\dot{x}}{x} - \frac{m\dot{x}}{a-x}$, and if $\dot{y} = 0$, $\frac{n}{x} = \frac{m}{a-x}$, that is, $AC : CG :: n : m$.

The same proposition may be derived from the former case when n and m are any integer numbers; for example, $AB \times BG^5$ is a *maximum* when AB is to BG as 1 to 5; because if BG be divided into 5 equal parts BC, CD, &c. then $AB \times BG^5 = 5 \times 5 \times 5 \times 5 \times 5 \times AB \times BC \times CD \times DE \times EF \times FG$, which is a *maximum* when $AB = BC = CD = DE = EF = FG$. If AG be divided into three parts AB, BD and DG, then $AB \times BD^n \times DG^m$ is a *maximum* when AB, BD and DG are to each other in the same proportion as the numbers 1, n and m respectively; because wherever we suppose the point B to be, $BD^n \times DG^m$ cannot be a *maximum*, (and consequently $AB \times BD^n \times DG^m$ is not a *maximum*,) unless $BD : DG :: n : m$; and wherever we suppose the point D to be, $AB \times BD^n$ cannot be a *maximum* unless $AB : BD :: 1 : n$. The continuation of those theorems is obvious; and this brief method of resolving several questions relating to *maxima* and *minima* that cannot be so easily reduced to the common rules, was mentioned in a letter to MARTIN FOLKES Esq; *Phil. transf.* N^o. 408. The following article gives another useful instance.

921. The radius AC and ark AF being given, let AF be divided into three parts, AE, EB and BF, let EM, EN and BR be the sines of the arks AE, EB and BF; then if $EM^n \times EN \times BR^m$ be a *maximum*, the tangents of the arks AE, EB and BF

FIG. 342.

A a a a a 2

BF

BF shall be in the same proportion as the numbers n , 1 and m . This follows from art. 910. because wherever we suppose the point B to be placed upon the ark FE, $EM^n \times EN$ is not a *maximum*, (art. 910.) unless the tangent of the ark AE be to the tangent of EB as n to 1; consequently the ark AB must be divided in this manner, that $BR^m \times EN \times EM^n$ may be a *maximum*. In like manner, wherever we suppose the point E to be taken upon the ark AB, $EN \times BR^m$ cannot be a *maximum*, unless the tangent of EB be to the tangent of BF as 1 to m ; and the ark FE must be divided in this manner that $EM^n \times EN \times BR^m$ may be a *maximum*. Therefore if $EM^n \times EN \times BR^m$ be a *maximum*, the tangent of AE must be to the tangent of EB as n to 1, and the tangent of EB to the tangent of BF as 1 to m ; that is the ark FA must be divided in such a manner in B and E that the tangents of AE, EB and BF may be in the same proportion to each other as the numbers n , 1 and m . If $n = m$, then $AE = BF$. The continuation of these theorems is likewise obvious. If a given ark be divided into any given number of parts whose sines are represented by a, b, c, d, e , &c. and $a^m \times b^n \times c^r \times d^s \times \&c.$ be a *maximum*, then the tangents of the respective parts must be in the same proportion as the indices, m, n, r, s , &c. and (because the sine of an ark is to the radius as the radius to the secant of the same ark) the product of the same powers of the respective secants of those parts is a *maximum*.

922. For an example of this, the force and direction of the wind being given, let it be required to find the most advantageous course of the ship and position of the sail, that the ship may be carried in a given direction, or removed from a given coast or right line, as fast as possible. Let AC represent the force and direction of the wind, CF the line from which the ship is to be carried as fast as possible, CB the course of the ship, and CE the position of the sail. Let AQ be parallel to CB, AP perpendicular to CE in P and PQ perpendicular to AQ in Q. Then the force by which the wind impells the ship in the direction CB at the beginning of the motion will be as $AP \times AQ$

$= EM \times \frac{EN}{CE}$; and the velocity of the ship (supposing it to be incomparably less than the velocity of the wind) shall be as $EM \times \sqrt{EN}$; which reduced to the direction BR perpendicular to CF is as $EM \times \sqrt{EN} \times BR$; and this last velocity is a *maximum* (by the last article) when the tangents of the arcs AE, EB and BF are in the same proportion as the numbers 1, $\frac{1}{2}$ and 1, or 2. 1 and 2. Let the radius $CE = a$, the tangent of AF be represented by b , the tangent of AE or BF by t , and the tangent of AB or FE by T . Because the arcs $AB + BF = AF$, it will easily appear that $t = a \times \frac{ab - aT}{aa + bT}$; and in the same manner, because $BF + BE = FE$, the tangent of BE ($= \frac{t}{2}$) $= a \times \frac{aT - at}{aa + Tt}$; whence $T = \frac{3aat}{2aa - tt}$; consequently $t^3 - 4btt - 5aat + 2baa = 0$; and b being given, t and T may be found by the resolution of this cubic equation.

923. If FCA be a right angle, then b is infinite and $2tt = aa$, or $t : a :: 1 : \sqrt{2}$ and $T : a :: \sqrt{2} : 1$; that is $ACB = FCE = 54^\circ 44'$; consequently if the velocity of the ship may be neglected as incomparably less than the velocity of the wind, the course ought to contain an angle of $54^\circ 44'$ and the sail an angle of $35^\circ 16'$ with the direction of the wind, that the ship may gain upon the wind as much as possible; and this is the case resolved by Mr. BERNOULLI *Manoeuvre des vaisseaux*, p. 50. &c. If the course CB and position of the sail CE is required that the ship may get away from the line AC as fast as possible, then we are to suppose ACF to be a continued right line, or $b = 0$, in which case $tt = 5aa$, or $t : a :: \sqrt{5} : 1$; consequently the angle ACE ought to be of $65^\circ 54'$ and ACB of $114^\circ 6'$. If the angle ACE be given, the tangent of ECB ought to be to the tangent of ECF as 2 to 1; and ECB is determined by a construction similar to that in art. 910. We have always supposed the sail to be a plane, and have abstracted from the lee-way of the ship, but shall not enter farther into this

this theory at present. Mr. RENAULT published an ingenious treatise on this subject in 1689; but some particulars in it have been corrected by Mr. HUYGENS and Mr. BERNOULLI. Several other mechanical problems may be resolved in the same manner as these we have considered.

924. In *Book I. chap. 13.* it was shown how many problems may be immediately reduced to equations that involve first fluxions only, which it has been usual to resolve first by equations that involve second or higher fluxions. But as that method is not always applicable, we shall give some examples of the method of reducing equations from second to first fluxions. Suppose x constant, and if the equation involve x , y and $\ddot{y}$, but if either x or y be wanting (of which kind are those which arise most commonly in the resolution of problems) it may be reduced to first fluxions, by introducing a new variable quantity z , and supposing it equal to $\frac{y}{x}$ or $\frac{x}{y}$. Suppose for example,

that $\dot{x}^2 + \dot{y}^2 = \frac{y\ddot{y}}{n}$, let $\dot{y} = zx$, and consequently $\ddot{y} = \dot{z}x$,

then $n\dot{x}^2 \times 1 + zz = y\dot{z}x$, or $n\dot{x} \times 1 + zz (=ny \times \frac{1+zz}{z}) = y\dot{z}$

therefore $\frac{2ny}{y} = \frac{2zz}{1+zz}$, and (by art. 740.) $y^{2n} = 1 + zz \times A$ or

$zz = \frac{y^{2n}}{A} - 1 = \frac{\dot{y}^2}{\dot{x}^2}$; consequently $\dot{x} = \frac{y\sqrt{A}}{\sqrt{y^{2n}-A}}$, where A

denotes an invariable quantity.

FIG. 343. 925. Let the point T move in the right line Aa , and the point M in the curve FM , so that the velocity of the point T may be to the velocity of the point M in the invariable ratio of n to 1, and the motion of M may be always in the direction MT or TM ; and let it be required to determine the curve FM . Let $AP = x$, $PM = y$, $FM = s$, $AT = t$; then $n\dot{s} = \dot{t}$ (because $t = x - \frac{yx}{y}$, or to $\frac{yx}{y} - x$, and $\dot{x}$ is supposed constant)

$$\frac{\dot{t} + y\ddot{y}x}{\dot{y}^2}$$

$\frac{\ddot{y}\dot{y}\dot{x}}{\dot{y}^2}$. Let $\dot{x} = z\dot{y}$, then $\dot{z}\dot{y} + z\ddot{y} = 0$, and $n\dot{y}\sqrt{1+zz} =$
 $\frac{\dot{y}\dot{y}\dot{z}\dot{x}}{z\dot{y}^2} = \pm y\dot{z}$, and $\frac{n\dot{y}}{y} = \frac{\dot{z}}{\sqrt{1+zz}}$; whence $Ay^n =$
 $\sqrt{1+zz} \pm z$, and $AAy^{2n} \mp 2Azy^n = 1$, or $2z = \frac{\dot{z}}{y} = \frac{\mp 1}{Ay^n}$
 $\pm Ay^n$, consequently $2\dot{x} = \frac{\mp \dot{y}}{Ay^n} \pm Ay^n \dot{y}$, and $2x = \frac{\mp 1}{1-n \times Ay^{n-1}}$
 $\frac{\pm Ay^n + 1}{n+1} + K$, where K denotes an invariable quantity.

If $n = \frac{1}{2}$, then $x = \frac{\mp \sqrt{y}}{A} \pm \frac{Ay\sqrt{y}}{3} + K$, and the curve is
 a parabola of the third order of lines. If $n = 1$, the curve is
 constructed by logarithms or the equilateral hyperbola. Let
 NDN be such an hyperbola described betwixt the asymptotes
 Ca and Cb, D a given point in the hyperbola, join CD, let
 NLM perpendicular to the asymptote Ca in L meet CD in K,
 and let $LM \times 2FD$ be always equal to the area DNK; then M
 shall be a point in the curve.

FIG. 344

926. An equation that involves second fluxions is sometimes
 easily reduced to first fluxions, by the common rules of the in-
 verse method, which were described in chap 2. and that the
 solution may be general enough, when any fluxion is supposed
 constant, a quantity compounded from it or from its powers and
 invariable quantities ought to be added to the equation. For
 example, let it be required to find the nature of the line
 in which the curvature is always as the ordinate, this being a
 figure by which several problems of different kinds are resolved.
 Let the ray of curvature be represented by R , and because

$R = \frac{\dot{y}\dot{s}^2}{s\ddot{x} - x\ddot{s}}$, suppose s constant, then $R = \frac{\dot{y}\dot{s}}{x\ddot{s}}$. In the fi-
 gure required R is inversely as the ordinate y ; consequently,
 a being an invariable quantity we may suppose $\frac{aa}{2y} = R = \frac{\dot{y}\dot{s}}{x\ddot{s}}$
 or

or $2yy\dot{s} = aax\ddot{x}$; and by finding the fluents, $yy\dot{s} = aax\dot{x} + K\dot{s}$ where K denotes an invariable quantity, and $K\dot{s}$ is added because $\dot{s}$ is supposed constant. If $K = 0$, then $\dot{s}^2 : \dot{x}^2 :: a^4 :$

$$y^4, \dot{y}^2 : \dot{x}^2 :: a^4 - y^4 : y^4, \text{ and consequently } \dot{x} = \sqrt{\frac{-y^4\dot{y} - a^4\dot{y}}{a^4 - y^4}}.$$

927. The celebrated Author who first resolved this as well as several other curious problems, after his account of this figure (which is commonly called the *elastic curve*), adds, *Ob graves causas suspicor curvæ nostræ constructionem a nullius sectionis conicæ seu quadratura seu rectificatione pendere, Act. Lips. 1694, p. 272.* But it is constructed by the rectification of the equilateral hyperbola; for if the base of a figure be always taken equal to the perpendicular from the center on the tangent of such an hyperbola, and the ordinate equal to the excess of the tangent terminated by that perpendicular above the ark intercepted betwixt the vertex of the hyperbola and the point of contact, then the figure shall be the elastic curve. Let AEZ be an equilateral hyperbola that has its center in S and vertex in A , let E be any point in the hyperbola, ET a tangent at E and SP perpendicular from the center S to the tangent at P ; upon SA take $SQ = SP$, and the ordinate QM always equal to the excess of the tangent EP above the ark AE of the hyperbola; then M shall be a point in the elastic curve AMB . For suppose $SA = a$, $SQ (= SP) = y$, $QM = x$, $SE = r$, $EP = z$, and the ark $AE = s$, then $r = \frac{aa}{y}$, $EP = z = \sqrt{rr - yy} = \frac{\sqrt{a^4 - y^4}}{y}$, $\dot{z} = \frac{-y^4\dot{y} - a^4\dot{y}}{yy\sqrt{a^4 - y^4}}$

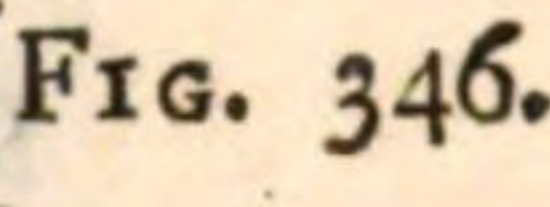
But $\dot{s} : \dot{r} :: r : z :: aa : \sqrt{a^4 - y^4}$ and $\dot{r} = \frac{-aay}{yy}$, consequently

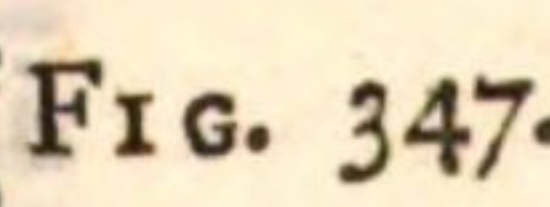
$$\dot{s} = \frac{-a^4\dot{y}}{yy\sqrt{a^4 - y^4}}, \text{ and } \dot{x} = \dot{z} - \dot{s} = \frac{-y^4\dot{y} - a^4\dot{y} + a^4\dot{y}}{yy\sqrt{a^4 - y^4}} =$$

$\frac{-y^4\dot{y}}{yy\sqrt{a^4 - y^4}}$, which is the equation for the common elastic

curve.

928. In

928. In general the equation for the elastic curve was $aa\dot{x} = yy\dot{s} - K\dot{s}$; consequently $\dot{x} = \dot{s} \times \frac{yy - K}{\sqrt{aa - K + yy} \times \sqrt{aa + K - yy}}$; and by comparing this fluxion with those described in art. 804. and 805. it will appear that the elastic curve may be constructed in all cases by the rectification of the conic sections. Let  FIG. 346. SA be half the transverse axis of the hyperbola AEH, SD half the second axis, upon DS take SF. $SA :: SA. SD$, and $Sb = AF$, describe the elliptic quadrant ARb, and E being any point in the hyperbola, SP perpendicular to the tangent EP in P, upon SA take $SQ = SP$, and let the ordinate QR meet the ellipse in R; then by taking QM upon QR equal to $\frac{AD^2}{2SD^2} \times \overline{EP - AE} + \frac{SD^2 - SA^2}{2SA \times SD} \times AR$, M shall be a point in the elastic curve: And the ray of curvature at any point M shall be equal to $\frac{AD^2}{4SQ}$, because in comparing those fluxions we suppose $aa - K = SD^2$ and $aa + K = SA^2$, or $2aa = SD^2 + SA^2 = AD^2$, and the ray of curvature was supposed equal to $\frac{aa}{2y} = \frac{AD^2}{4SQ}$.

929. Let SA be incomparably less than SD, then because  FIG. 347. $SD : SA :: SA : SF$, we may suppose SF to vanish, ARb to be a quadrant of a circle, and $EP - AE$ to vanish; consequently $QM = \frac{SD}{2SA} \times AR$ and $SB = \frac{SD}{2SA} \times ARb =$ (if the ratio of m to 1 denote that of the circumference to the diameter) $\frac{m}{4} \times SD$: And the elastic curve in this case will represent the figure which a musical chord BAC assumes in its small vibrations, by the converse of art. 569, the tension of the chord being every where equal. Let P represent this tension, or a weight equivalent to it, n a section of the chord perpendicular to its length, R the ray of curvature at any point M, and V the force by which the motion of any point at M towards AC is accelerated

ed, while the chord returns to its natural state; then by art. 561. the tension would be equal to the weight of a chord of the same thickness of the length R , if the gravity was equal

to V ; that is, $P = nRV$, or $V = \frac{P}{nR} = \frac{P}{n} \times \frac{4SQ}{AD^2} =$ (because SD is to AD nearly in a ratio of equality) $\frac{P}{n} \times \frac{4SQ}{SD^2} =$

$\frac{mmP}{n} \times \frac{SQ}{4SB^2}$. If we suppose with Dr. TAYLOR, N to represent the weight of the chord, L its length, g the force of gravity, D the length of a given pendulum, $SA = a$, SQ or $MN = y$; then, because $N = nLg$, or $n = \frac{N}{Lg}$, and $L = 2SB$,

it follows that $V = \frac{mmPg y}{NL}$. Because V is as y the distance of M from AC , the vibrations of the chord are similar to those of a pendulum; and the time in which M describes MN is to the time in which the pendulum D performs half a vibration as $\sqrt{\frac{y}{V}}$ to $\sqrt{\frac{D}{g}}$, or as $\frac{\sqrt{LN}}{m\sqrt{P}}$ to $\sqrt{D}$; consequently the number of vibrations made by the chord, while the pendulum vibrates once, is expressed by $m \times \sqrt{\frac{DP}{LN}}$; which

is Dr. TAYLOR's theorem, and serves for determining the number of vibrations made in a given time by any given chord that is extended by a given weight; or for comparing the number of vibrations made by different chords in equal times, upon which their tone depends. Thus if the weight P be the same, the number of vibrations is as $\frac{1}{\sqrt{LN}}$; and when the chords are of the same kind (or N is as L) the vibrations are as $\frac{1}{L}$. If the chord be given, the number of vibrations is as $\sqrt{P}$. The ratio of m to 1, or of the circumference to the diameter, enters the expression of the number of the vibrations of the chord; because the ratio of $2SB$ the length of the chord to the ray of curvature involves it; and there seems to be a difference

ference in this respect betwixt the theorems by which the vibrations of a musical chord, and these which are produced in the air by organ-pipes, or other wind-instruments, are to be determined.

930. Because the elastic curve is defined by the equation,

(art. 926.) $yy\dot{s} - K\dot{s} = aax$, it follows that $\dot{s} = \frac{\mp aay}{\sqrt{a^4 - yy - K^2}}$

$= \frac{\mp aay}{\sqrt{aa + K - yy} \times \sqrt{aa - K + yy}}$. Let this fluxion be compared with

that in art. 805. of which we found the fluent to be $\frac{SD}{SA} \times AR$ FIG. 346.

+ AE - EP; and it will appear, by supposing $aa + K = SA^2$, and $aa - K = SD^2$, that AM the ark of the elastic curve

is equal to $\frac{AD^2}{2SD^2} \times \frac{SD}{SA} \times AR + AE - EP$. Therefore the

figures that have been constructed by the rectification of the elastic curve, may be constructed by the rectification of the hyperbola and ellipsis; particularly the curve along which if a heavy body moved, it would recede equally in equal times from a given point, which Mr. LEIBNITZ constructed by the rectification of a geometrical curve of a higher kind than the conic sections, and Mr. JAMES BERNOULLI by the elastic curve, *Act. Lips.* 1694. p. 272. 277. 338. 370. &c. The flu-

ents of $\frac{z^2\dot{z}}{\sqrt{z^4 - a^4}}$, $\frac{a^2\dot{z}}{\sqrt{z^4 - a^4}}$, $\frac{a^2\dot{z}}{\sqrt{a^4 - z^4}}$, and $\frac{z^2\dot{z}}{\sqrt{a^4 - z^4}}$ (which are

mentioned *ibid.* p. 338. where it is said of the first only, that it may be assigned by the rectification of the hyperbola) are all assignable by the rectification of the equilateral hyperbola, and of the ellipsis, whose excentricity is equal to the second axis.

Let AE and AR be such an hyperbola and ellipsis, $SA = a$, FIG. 348.

and $SE = z$, then the F. $\frac{z^2\dot{z}}{\sqrt{z^4 - a^4}} = AE$, and the fluent of

$\frac{a^2\dot{z}}{\sqrt{z^4 - a^4}} = AR + AE - EP$. If SP be perpendicular from

B b b b b 2

the

the center S on the tangent at E in P, $SA = a$ and $SP = z$,

then the F. $\frac{-a^2 \dot{z}}{\sqrt{a^4 - z^4}} = AR + AE - EP$, and the F. $\frac{-z^2 \dot{z}}{\sqrt{a^4 - z^4}}$

$= EP - AE$, as appears from art. 799. and 802. Fluents of other forms may be assigned by the rectification of the conic sections by art. 804. and 805.

931. It may be worth while to show here how the same easy method which was described in *chap. 13. Book I.* for determining, by first fluxions only, the nature of the lines of swiftest descent, of the figures that amongst all those of equal perimeters produce *maxima* and *minima*, and of that which generates the solid of least resistance, serves with equal facility and evidence for discovering the equation of the curve when other limitations are added in the problem; as when it is required to find the solid, which amongst all those of equal capacities, and that are bounded by equal surfaces, meets with the least resistance in a fluid. The fundamental *lemma* (demonstrated in **FIG. 349.** art. 572. and 592.) is that if AK be given, KE be perpendicular to AK , a and u denote any given or invariable quantities, then $AE \times a - KE \times u$ (or $\frac{AE}{u} - \frac{KE}{a}$) is a *minimum* when $KE : AE :: u : a$, or $a \times KE = u \times AE$. Let the base $FP = x$, the ordinate $PA = y$, the ark $GA = s$, $AK = y$, and if AE the tangent at A meet KE parallel to the base in E , then $AE = s$ and $KE = x$: And it follows from the *lemma*, that if V and u represent any quantities compounded from the powers of y (so as to be of the same value when y is the same) and if y be given, then $V\dot{s} - u\dot{x}$ and $\frac{s}{u} - \frac{x}{V}$ are *minima* when $V\dot{x} = u\dot{s}$. From this it follows (as in art. 576. and 593.) that if GAD be the whole curve, and DH the difference of the ordinates at G and D be given, then the F. $V\dot{s} - F. u\dot{x}$, or the F. $\frac{s}{u} - F. \frac{x}{V}$ shall be a *minimum* when the nature of the figure is defined by the equation $V\dot{x} = u\dot{s}$. Therefore supposing
this

this to be the equation of the curve, and DH to be given, if the fluent of $V\dot{s}$ be also given, then the $F. u\dot{x}$ shall be a *maximum*; or if the latter fluent be given, then the former shall be a *minimum*: And if the fluent of $\frac{\dot{s}}{u}$ be given, the $F. \frac{\dot{x}}{V}$ shall be a *maximum*; or if the fluent of $\frac{\dot{x}}{V}$ be given, the fluent of $\frac{\dot{s}}{u}$ shall be a *minimum*. It appears likewise (as in art 595.) that if DH with the base FC or GH be given, and the fluent of $V\dot{s}$ be given or invariable, then the $F. u\dot{x}$ will be a *maximum* or *minimum* when the equation of the curve is $V\dot{x} = \overline{e + u \times s}$, where e denotes an invariable quantity that may be positive, or negative, or vanish.

932. Suppose therefore $V = A + B\dot{y} + C\ddot{y} + D\dot{y}^3 + \&c.$ and $u = a + b\dot{y} + c\dot{y}^2 + d\dot{y}^3 + \&c.$ where $A, B, C, \&c.$ and $a, b, c, \&c.$ denote any invariable coefficients that may be positive or negative, any of which may be supposed to vanish; and the fluent of $V\dot{s} - u\dot{x}$, that is, of $\dot{s} \times A + B\dot{y} + C\ddot{y} + \&c. - \dot{x} \times a + b\dot{y} + c\ddot{y} + \&c.$ shall be a *minimum*, when the equation of the figure is $\dot{x} \times A + B\dot{y} + C\ddot{y} + \&c. = \dot{s} \times a + b\dot{y} + c\ddot{y} + \&c.$ the ordinate DH being given. Therefore if the fluent of $\dot{s} \times A + B\dot{y} + C\ddot{y} + \&c.$ be also given, the fluent of $\dot{x} \times a + b\dot{y} + c\ddot{y} + \&c.$ shall be a *maximum*; or if the latter be given, the former shall be a *minimum*: And if the base FC or GH be given with DH and the $F. \dot{s} \times A + B\dot{y} + C\ddot{y} + \&c.$ then the $F. \dot{x} \times a + b\dot{y} + c\ddot{y} + \&c.$ shall be a *maximum* or *minimum* when $\dot{x} \times A + B\dot{y} + C\ddot{y} + \&c. = \dot{s} \times \overline{e + A + B\dot{y} + C\ddot{y} + \&c.}$ Of which theorem it is an obvious, but a particular case only, that if the nature of the figure be defined by the last equation, and GH, DH, with the fluents of $A\dot{s}$, $B\dot{y}\dot{s}$, $C\ddot{y}\dot{s}$, $\&c.$ $b\dot{y}\dot{x}$, $c\ddot{y}\dot{x}$, $d\dot{y}^3\dot{x}$, $\&c.$ be all given or invariable, one only excepted, this last fluent shall be either a *maximum* or *minimum*.

933. For example, the points G and D being given, if the peri-

perimeter GAD (or the F. $\dot{A}s$) be also given, the area FGADC (or the F. $\dot{y}x$) is a *maximum* or *minimum* when $\dot{A}x = \dot{s} \times e \mp \overline{A + By}$, that is, when GAD is an ark of a circle. If the surface generated by the ark GAD about the axis FC (or the F. $\dot{B}ys$) be given, then the solid generated by the figure FGADC about the same axis (or the F. $\dot{c}yyx$) is a *maximum* or *minimum* when $\dot{B}yx = \dot{s} \times e \mp \overline{cyy}$; and when $e = 0$, this is again a circle. If the perimeter GAD (or the F. $\dot{A}s$) be given, the solid generated by FGADC about the axis FC is a *maximum* or *minimum* when $\dot{A}x = \overline{e + cyy} \times \dot{s}$, and GAD is the elastic curve, which was constructed by the arks of conic sections in art. 928. If the F. $\dot{s} \times \overline{A + By}$ be given, then the fluent of $\dot{x} \times \overline{a + by + cyy}$ is a *maximum* or *minimum* when $\dot{x} \times \overline{A + By} = \dot{s} \times e \mp \overline{a + by + cyy}$. And it is no more but a particular case of this theorem that the same equation comprehends that of the figure when the points G, D, with the surface generated by GAD about FC and the area FGADC, are given or invariable, and the solid generated by this area about the axis FC is a *maximum* or *minimum*. For since, by the supposition, the fluents of $\dot{s} \times \overline{A + By}$ with the F. $\dot{a}x$ and F. $\dot{b}yx$ are given, so that the F. $\dot{c}yyx$ alone is variable, and the fluent of F. $\dot{x} \times \overline{a + by + cyy}$ is a *maximum* or *minimum*, it is manifest that the F. $\dot{c}yyx$ is a *maximum* or *minimum*. Nor is there any occasion, in order to obtain such equations, to have recourse to higher fluxions, or to resolve the element of the curve into a number of infinitesimal parts. Other examples may be derived in the same manner at pleasure.

934. The same method is extended to several other sorts of problems by art. 605. Let V be now compounded from the powers of s and y as well as from the powers of y and invariable quantities. For example, let $V = \frac{Ky^n}{s^n} + A + By + Cy^2 + Dy^3 + \&c.$ where K is supposed to be compounded from the powers of y and invariable quantities, and $u = a + by + cyy$

$cyy + \&c.$ as formerly. Then it appears, as in article 605, that $V\dot{s} \mp ux$ shall be a *minimum* when $1-n \times$

$$\frac{Ky^n \dot{x}}{\dot{s}^n} + A\dot{x} + B\dot{y}\dot{x} + Cy^2 \dot{x} + \&c. = \mp \dot{s}u =$$

$\mp \dot{s} \times \overline{a + by + cyy + \&c.}$ and by substituting 3 for n this equation serves for resolving the problems that may be proposed concerning the solid of least resistance. For supposing the solid that is generated by the revolution of the figure FGADC to move in a fluid with a given velocity and in the direction of the axis CF; then, according to the common doctrine of the action of the particles of fluids on bodies, (or if the fluid be rare as Sir ISAAC NEWTON supposes,) the resistance of the conical surface generated by the tangent AE will be ultimately

FIG. 350.

$$\text{ly as } PA \times AK \times \frac{AK^2}{AE^2} = \frac{yy^3}{\dot{s}^2} = \frac{yy^3}{\dot{s}^3} \times \dot{s}, \text{ and } 2yy^3 \dot{x} - \dot{x} \dot{s}$$

$\times A + B\dot{y} + Cy\dot{y} + \&c. = \dot{s}^4 \times \overline{a + by + cyy + \&c.}$ is the equation for the curve that generates the solid of least resistance, when the points G and D with the fluents of $A\dot{s}$, $B\dot{y}\dot{s}$, $Cy\dot{y}\dot{s}$ and the F. $\dot{x} \times \overline{a + by + cyy + \&c.}$ are supposed to be given or invariable. Thus if the points G and D only are given, the equation is $2yy^3 \dot{x} = a\dot{s}^4$, as Sir ISAAC NEWTON found. If the solid of the least resistance is required amongst all the solids of equal capacity, the equation is $2yy^3 \dot{x} = a\dot{s}^4 + cy^2 \dot{s}^4$. If the solids are supposed to be bounded by equal surfaces, the equation for the figure which generates the solid of least resistance is $2yy^3 \dot{x} - B\dot{y}\dot{x} \dot{s}^3 = a\dot{s}^4$. If the solid is to have the least resistance of all those that have equal capacity, and are terminated by equal surfaces, the equation is $2yy^3 \dot{x} - B\dot{y}\dot{x} \dot{s}^3 = a\dot{s}^4 + cy^2 \dot{s}^4$; and in like manner the equation is found, when other limitations that relate to the perimeter GAD, area FGADC, &c. are superadded.

935. Because the theorems proposed in art. 563. and explained in the subsequent articles, are of more general use, it may be proper to give one example of the manner of applying them for

for discovering the equation of the figure required. Let u the velocity acquired at any point A be as $Ax^n + By^m \times x^k y^l$ and the equation of the line of swiftest descent be required. Let OA the ray of curvature at A be considered as given in position, and supposing the point O to remain, let A move in the right line OA and AP be always perpendicular to FC in P ; let $OA = q$, $FP = x$ and $AP = y$, then if OA meet FC in I , $-\dot{x} : \dot{q} :: PI : IA :: \dot{y} : \dot{s}$, and $\dot{y} : \dot{q} :: PA : IA :: \dot{x} : \dot{s}$. But by the theorem in art. 565. OA and u increase proportionally while the point A is supposed to move in the right line OA , that is, $\frac{\dot{q}}{q} = \frac{\dot{u}}{u} = \frac{nAx^{n-1}\dot{x} + mBy^{m-1}\dot{y}}{Ax^n + By^m} + \frac{k\dot{x}}{x} + \frac{l\dot{y}}{y}$.

Hence by substituting $-\frac{\dot{q}\dot{y}}{\dot{s}}$ for $\dot{x}$, and $\frac{\dot{q}\dot{x}}{\dot{s}}$ for $\dot{y}$, then dividing by $\dot{q}$, and substituting for the ray of curvature q its value $\frac{\dot{s}}{-\dot{x}\dot{y}}$, x being supposed constant, it follows that $\frac{\dot{y}\dot{x}}{\dot{s}^2} = \frac{nAx^{n-1}\dot{y} - mBy^{m-1}\dot{x}}{Ax^n + By^m} + \frac{k\dot{y}}{x} + \frac{l\dot{x}}{y}$. If it be required that the

curve shall be described in less time than any other of an equal perimeter, the equation may be found by the third general principle described in art. 563.

936. The preceding examples may serve to show the extensive usefulness of the method of fluxions in Geometry and the various parts of Philosophy. In the account we gave of this doctrine in the first Book, we supposed with Sir ISAAC NEWTON quantities to be generated by motion, and considered the fluxion of a quantity as the velocity or measure of this motion. Some propositions however were demonstrated (as prop. 20. and 32.) without making use of fluxions; and several other theories described in this and the preceding Book may be likewise established in a manner independent of the notion of a fluxion

xion. Thus, it is easily demonstrated from art. 710. that, sup-
 posing n to be any integer and positive number, if the area FIG. 352.
 upon the base AP or x be always equal to x^n , then the ordinate
 PM or y shall be always equal to nx^{n-1} . For let o represent
 Pp any increment of the base x ; and because x and y increase
 together, PM $\times$ Pp, or $y \times o$, shall be less than PMmp =
 $\overline{x + o}^n - x^n$ the simultaneous increment of the area, which (by
 substituting $x + o$ for E, and x for F, in art. 710.) is less than
 $n \times \overline{x + o}^{n-1} \times o$; consequently y is less than $n \times \overline{x + o}^{n-1}$.
 In the same manner it appears that PM $\times$ Pp, or $y \times o$, is greater
 than PM $\mu\mu$ = $x^n - \overline{x - o}^n$, which, by the same article, is
 greater than $no \times \overline{x - o}^{n-1}$; consequently the ordinate y is
 greater than $n \times \overline{x - o}^{n-1}$. But if y be said to be greater than

nx^{n-1} , suppose $y = nx^{n-1} + r$, and $o = \overline{x^{n-1} + \frac{r}{n}}^{1/n} - x$,
 or $\overline{x + o}^{n-1} = x^{n-1} + \frac{r}{n}$, then $y = nx^{n-1} + r = n \times$
 $\overline{x + o}^{n-1}$; the contrary of which has been demonstrated.
 And if y be said to be less than nx^{n-1} , suppose $y = nx^{n-1}$
 $- r$, and $o = x - \overline{x^{n-1} - \frac{r}{n}}^{1/n}$ or $\overline{x - o}^{n-1} = x^{n-1} -$
 $\frac{r}{n}$, then $y = n \times \overline{x - o}^{n-1}$, against what has been demon-

strated. Therefore $y = nx^{n-1}$. I intended to have subjoined
 demonstrations of this kind, of some other theorems;
 but this seems to be unnecessary, after what has been shown
 at so great length in the first book and the first chapter of this
 book, for demonstrating the evidence of this method. Some-
 times we have spoke of infinites in this chapter, in the u-
 sual stile of writers on this subject; but we took no greater
 liberty in making use of such expressions than is allowed to
 authors in the inferior parts of these sciences, particularly to

C c c c c

such

such as treat of Trigonometry; who while they assign a tangent and secant to every ark, and find that no finite tangent or secant can belong to the quadrant, therefore mark it *infinite* in their tables. In the same sense y or $\dot{y}$ are in certain cases supposed to become infinite. But we pretend to draw no conclusions concerning infinities from the use of such concise and convenient expressions, nor inferences of any kind, but such as may be otherwise justified by unexceptionable evidence.

937. In this doctrine, when the velocity of a motion is determined, it is always with relation to the velocity of some other motion; and when we enquire at what rate the ordinate, for example, increases or decreases, it is always in relation to the base, or some other magnitude, with which it is compared. It is only relative space and motion we have occasion to consider in this method, than which no sort of quantities seem to be more clearly conceived by us.

F I N I S.

ERRATA

Fig. 319. N.2. Art. 866.

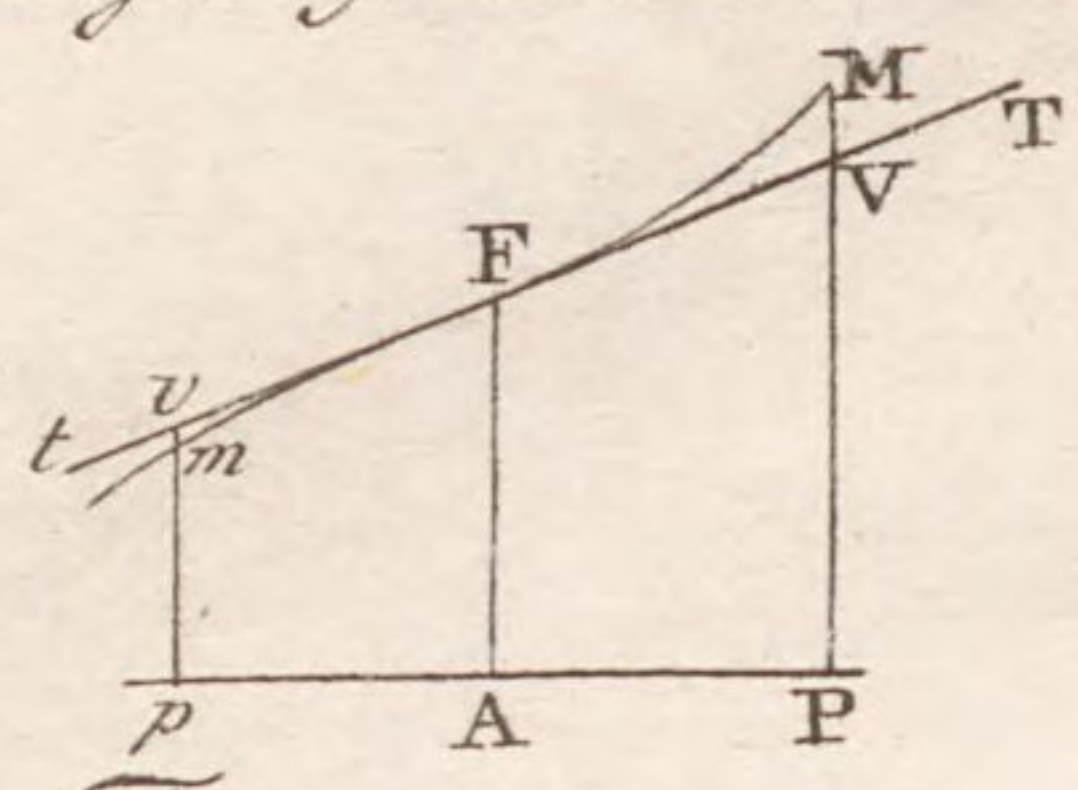


Fig. 321.

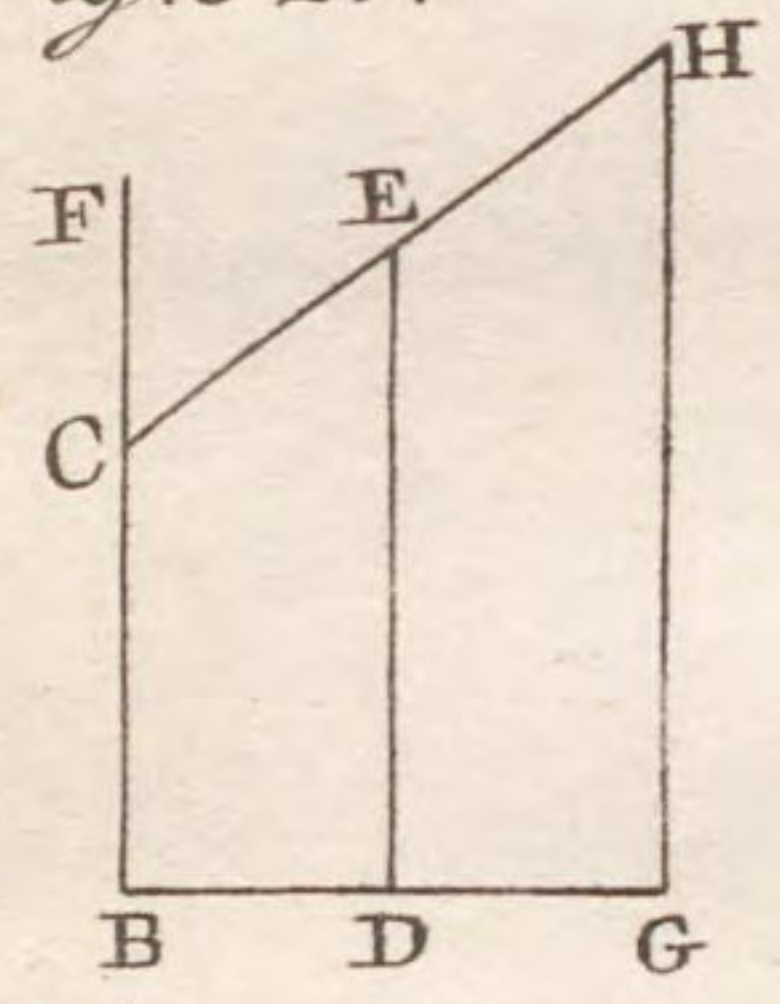


Fig. 323.

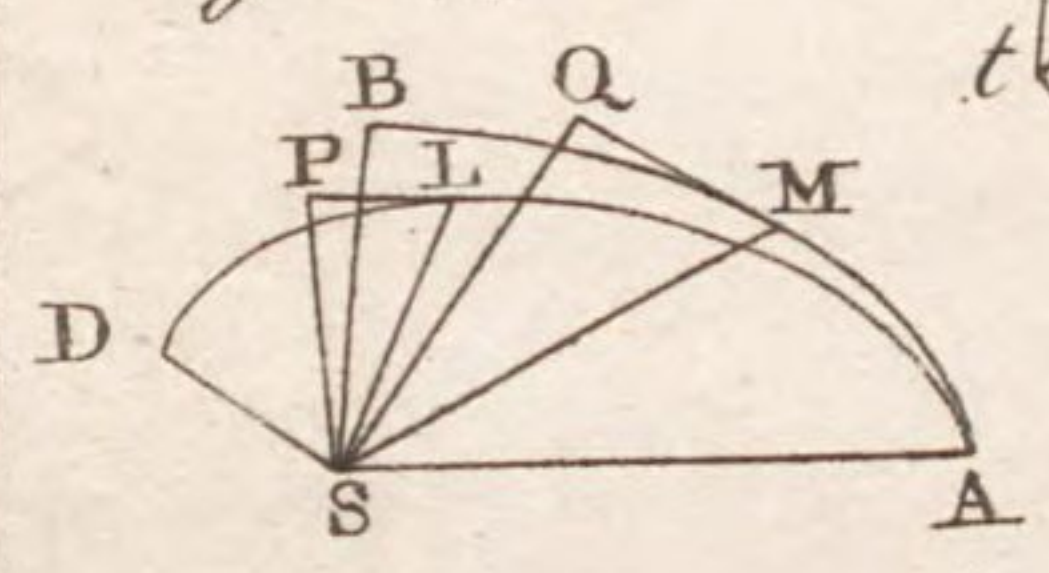


Fig. 319. N.1.

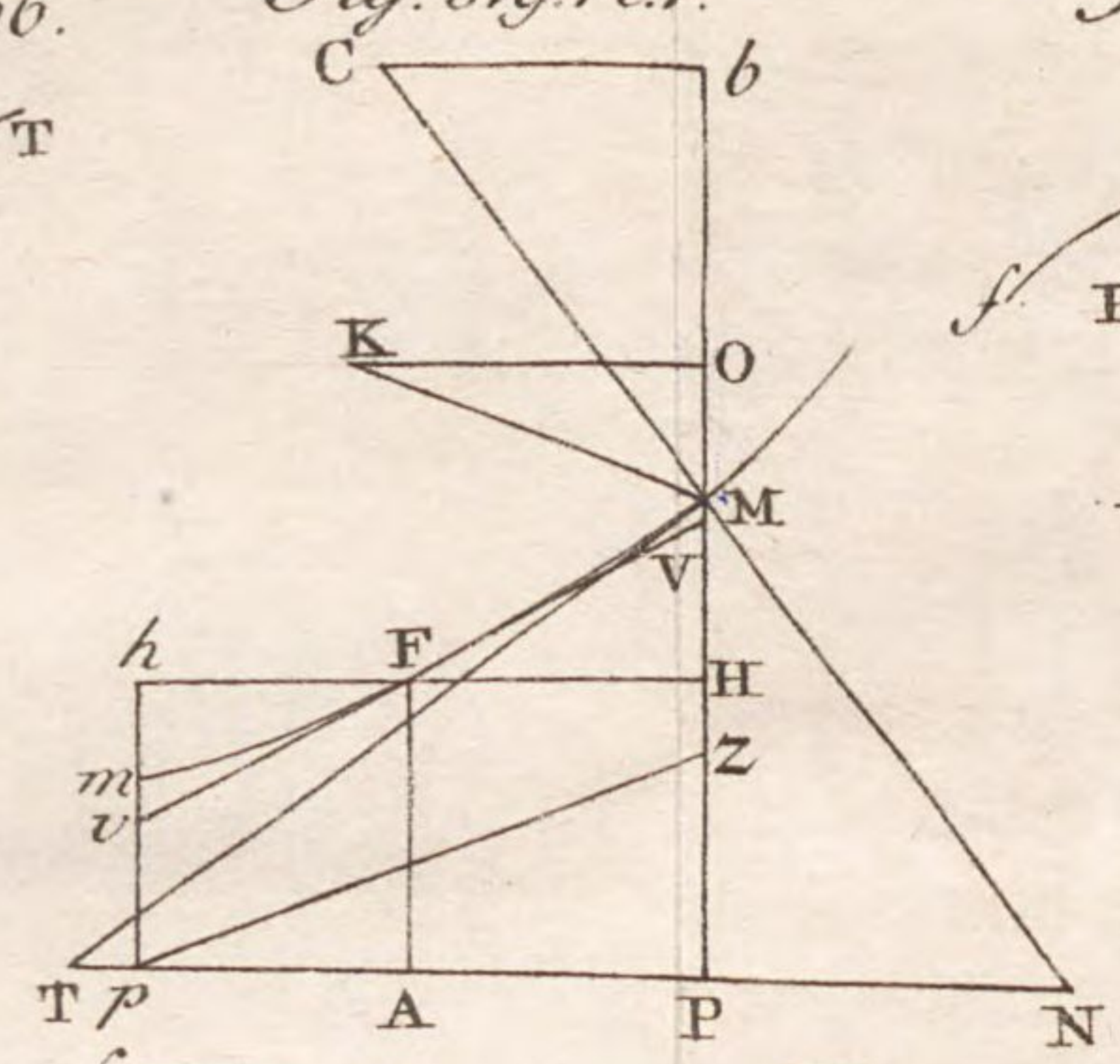


Fig. 322. N.1.

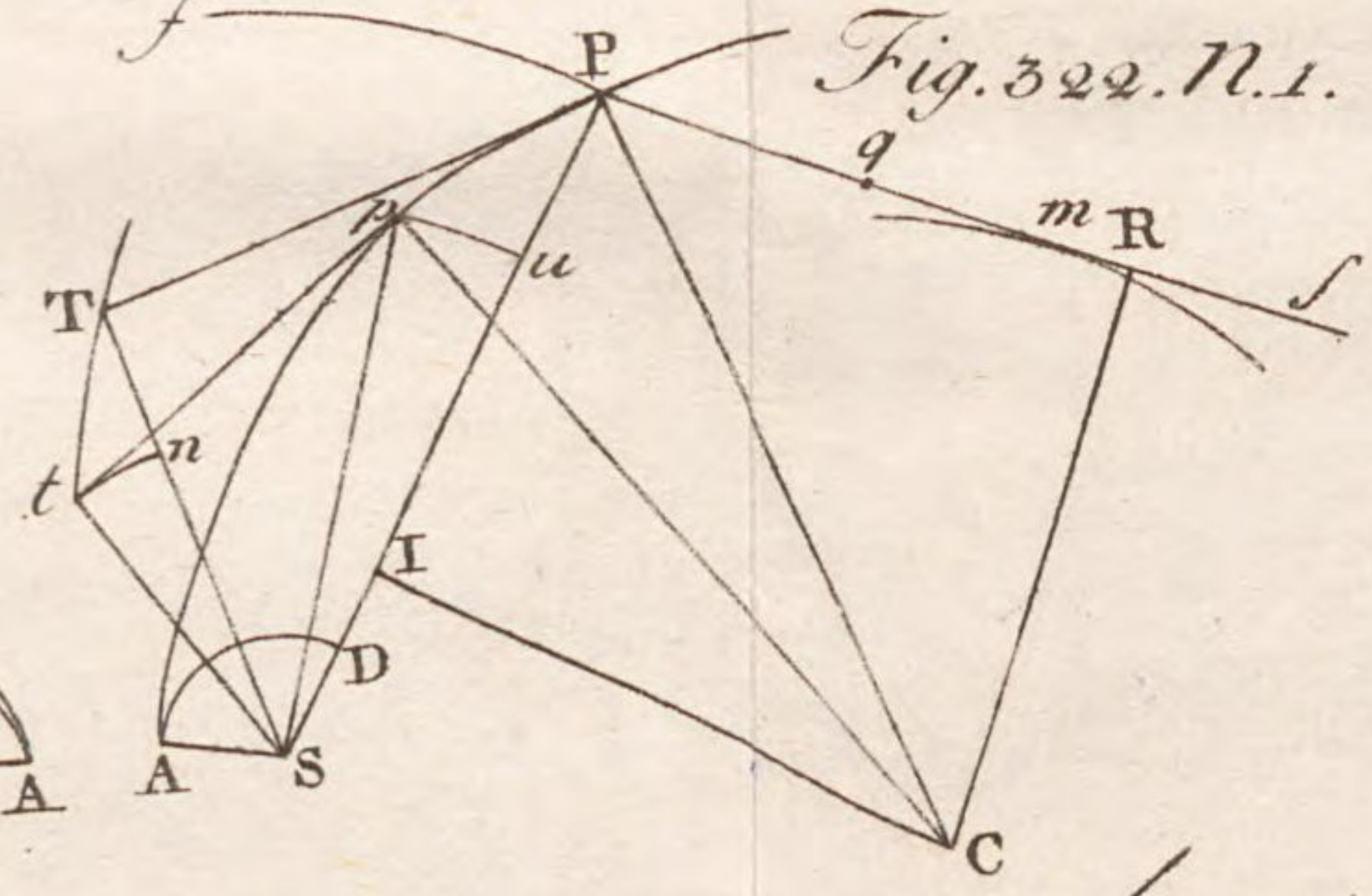
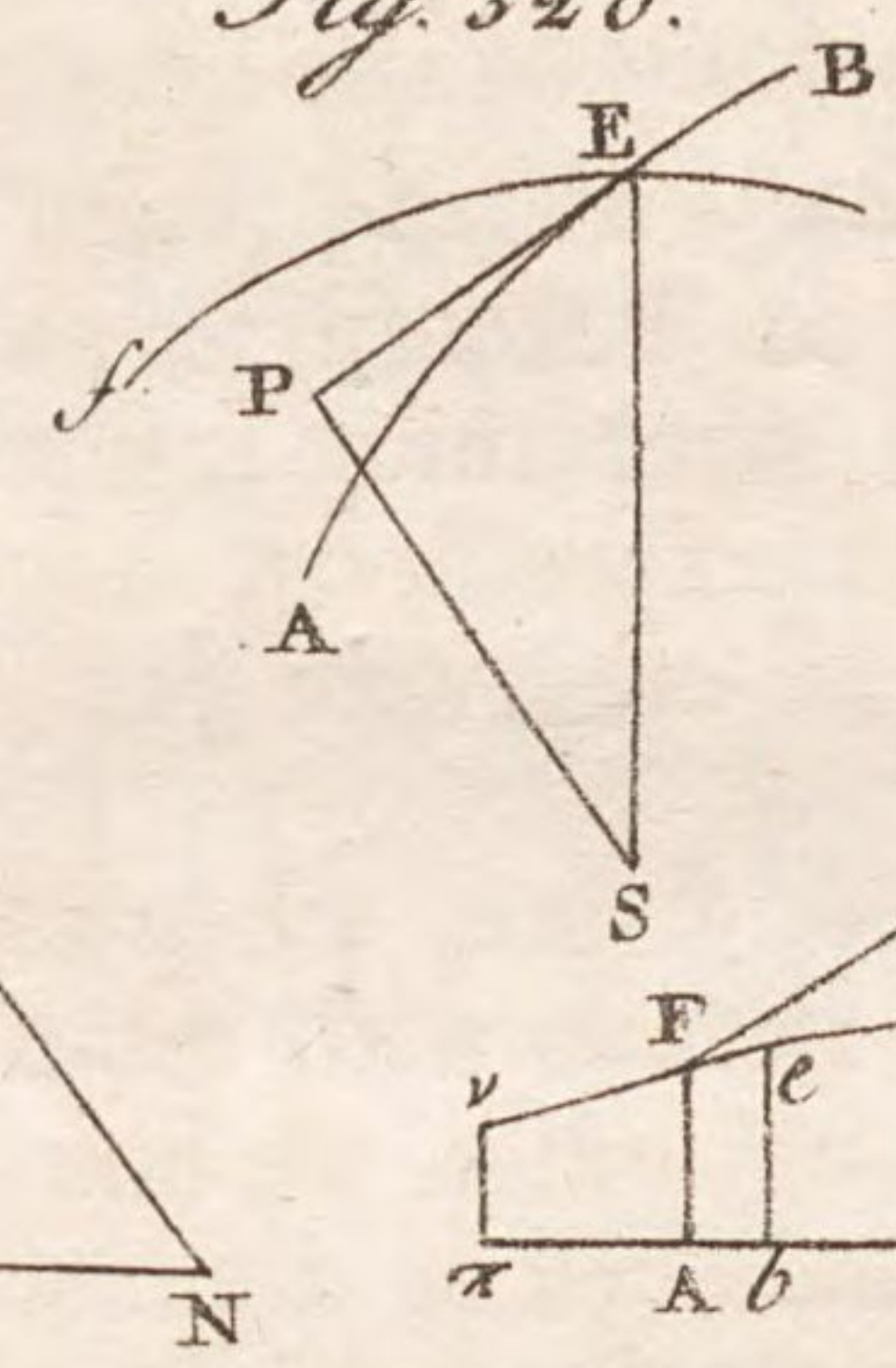


Fig. 320.



Pl. XXXVII. N.2

Fig. 318.

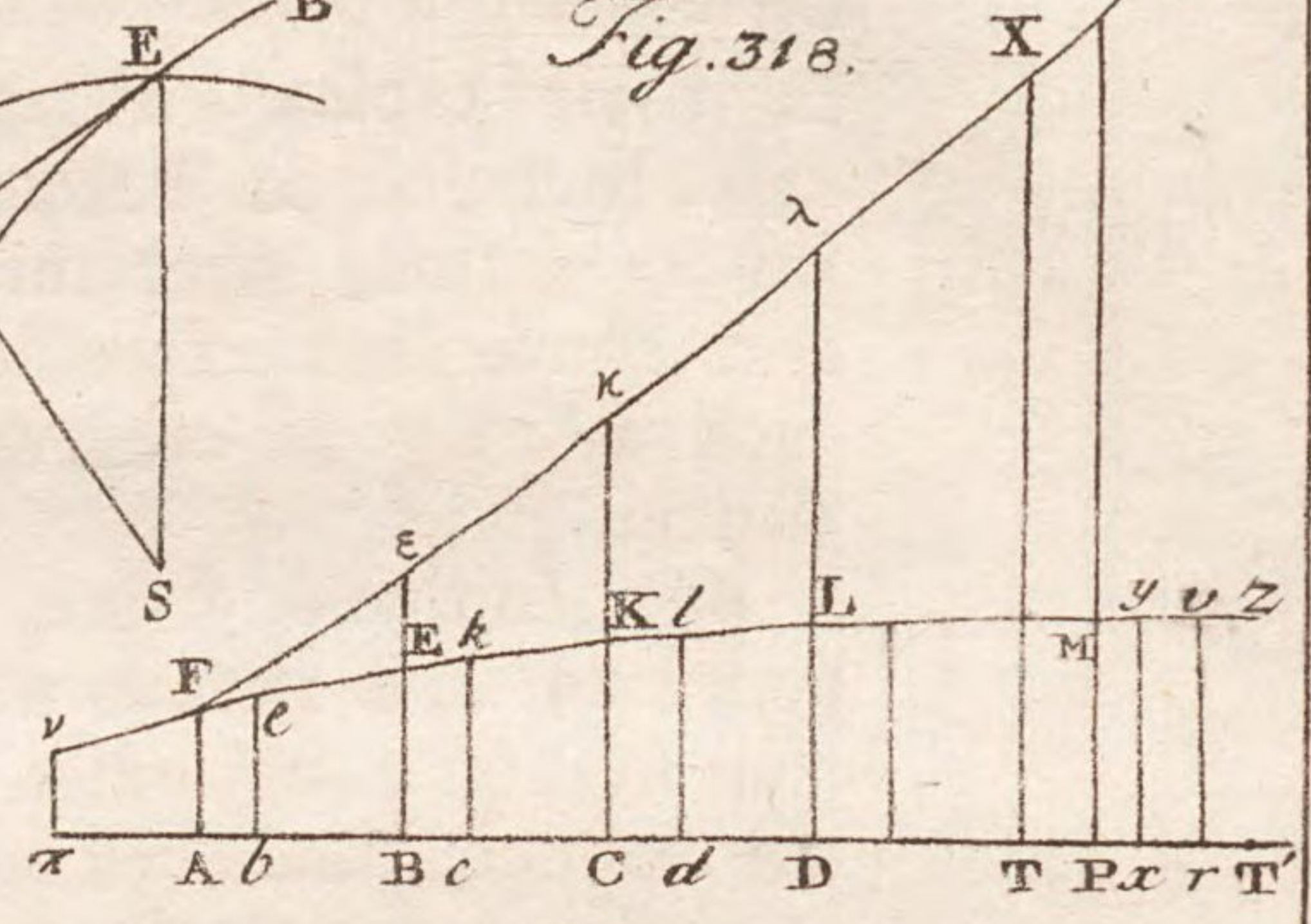


Fig. 322. N.2.

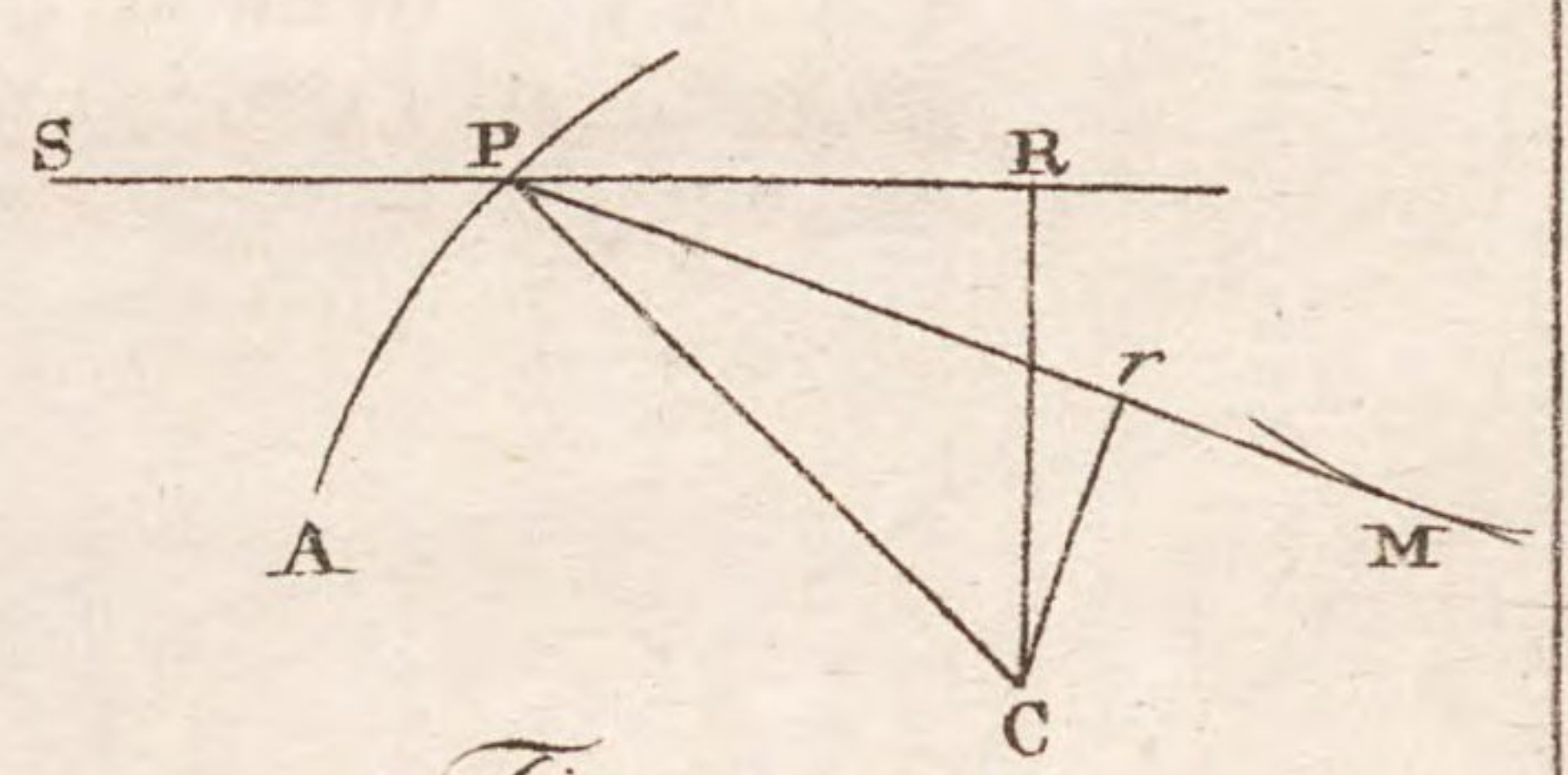


Fig. 325.

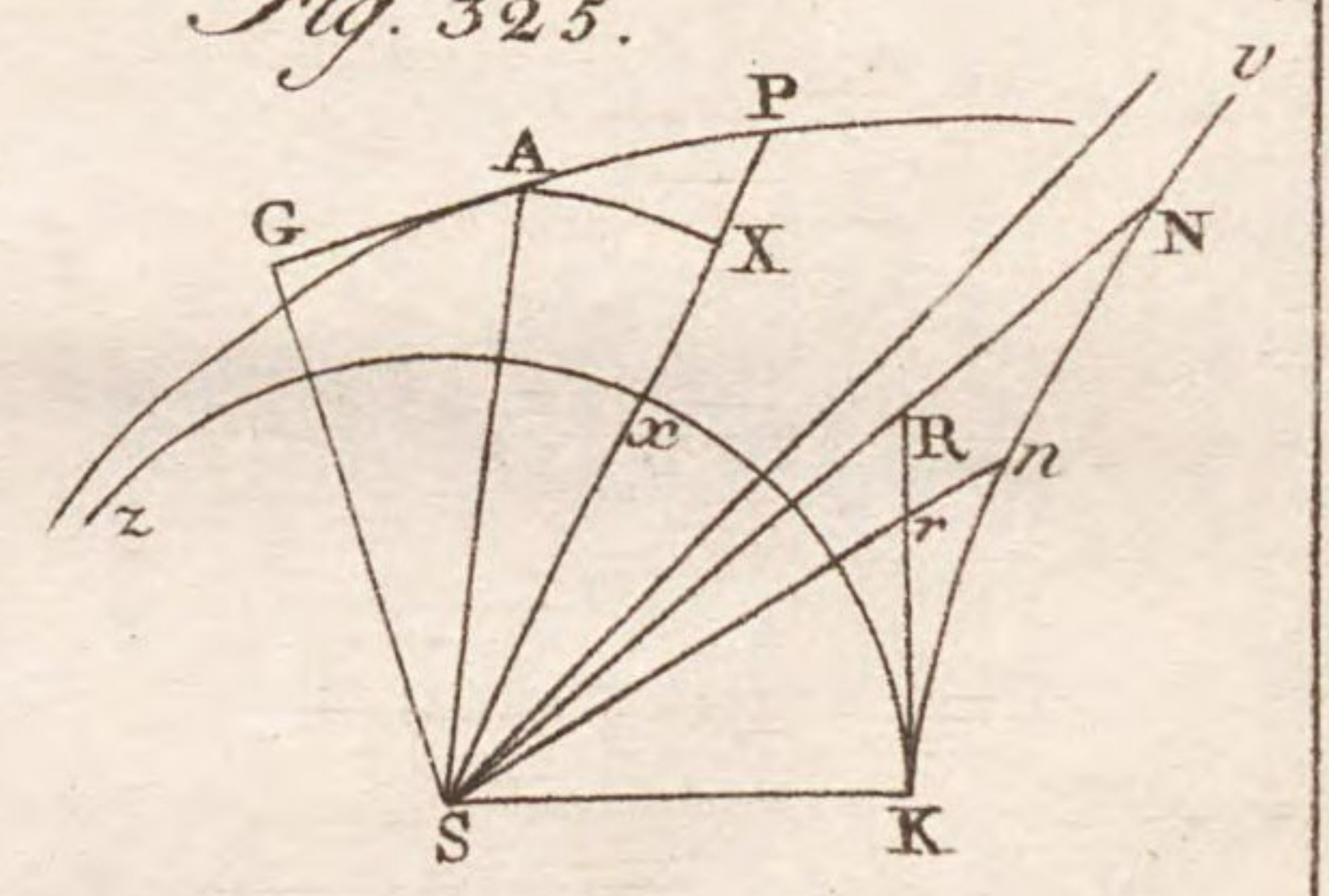


Fig. 324.

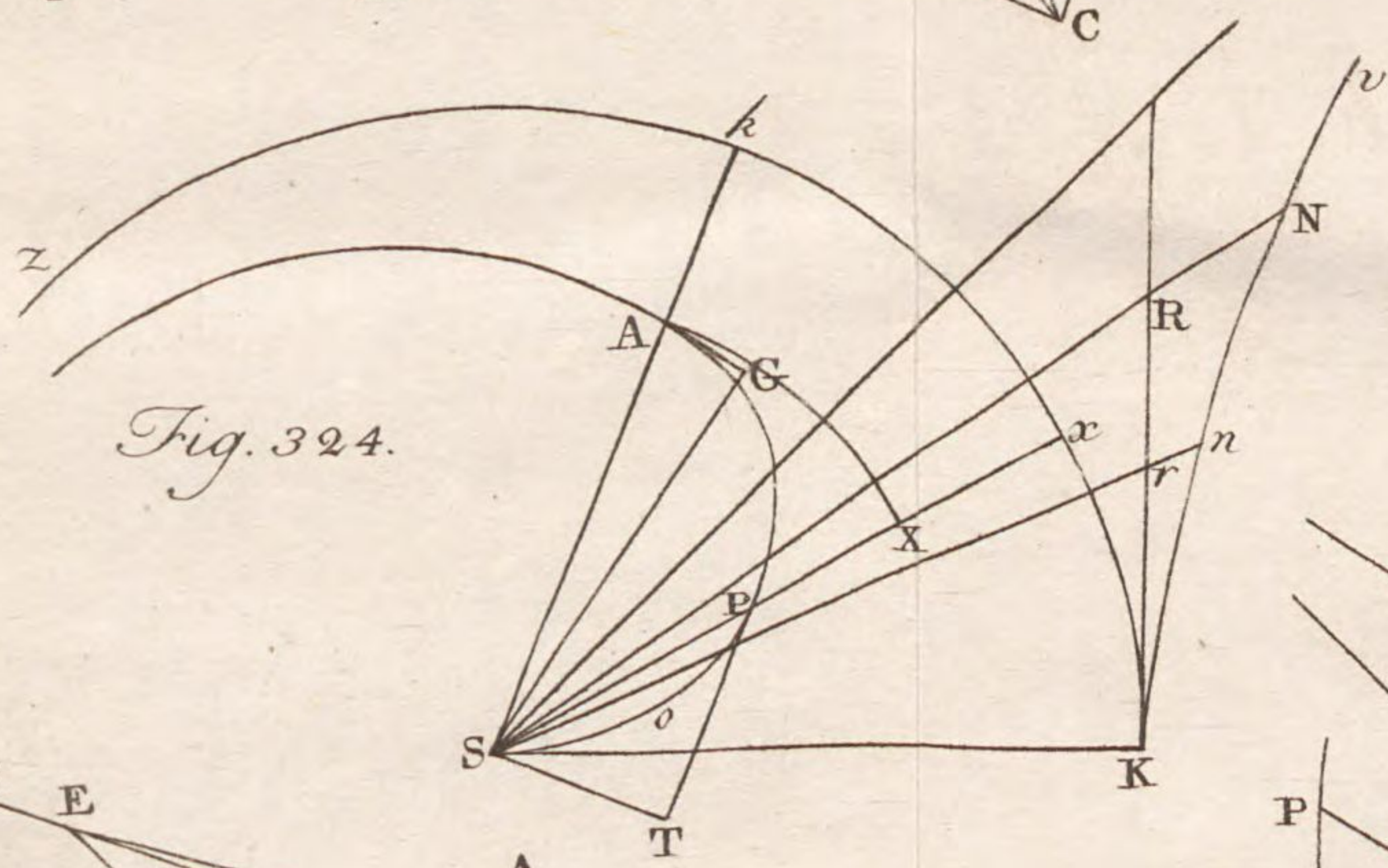


Fig. 326. N.1.

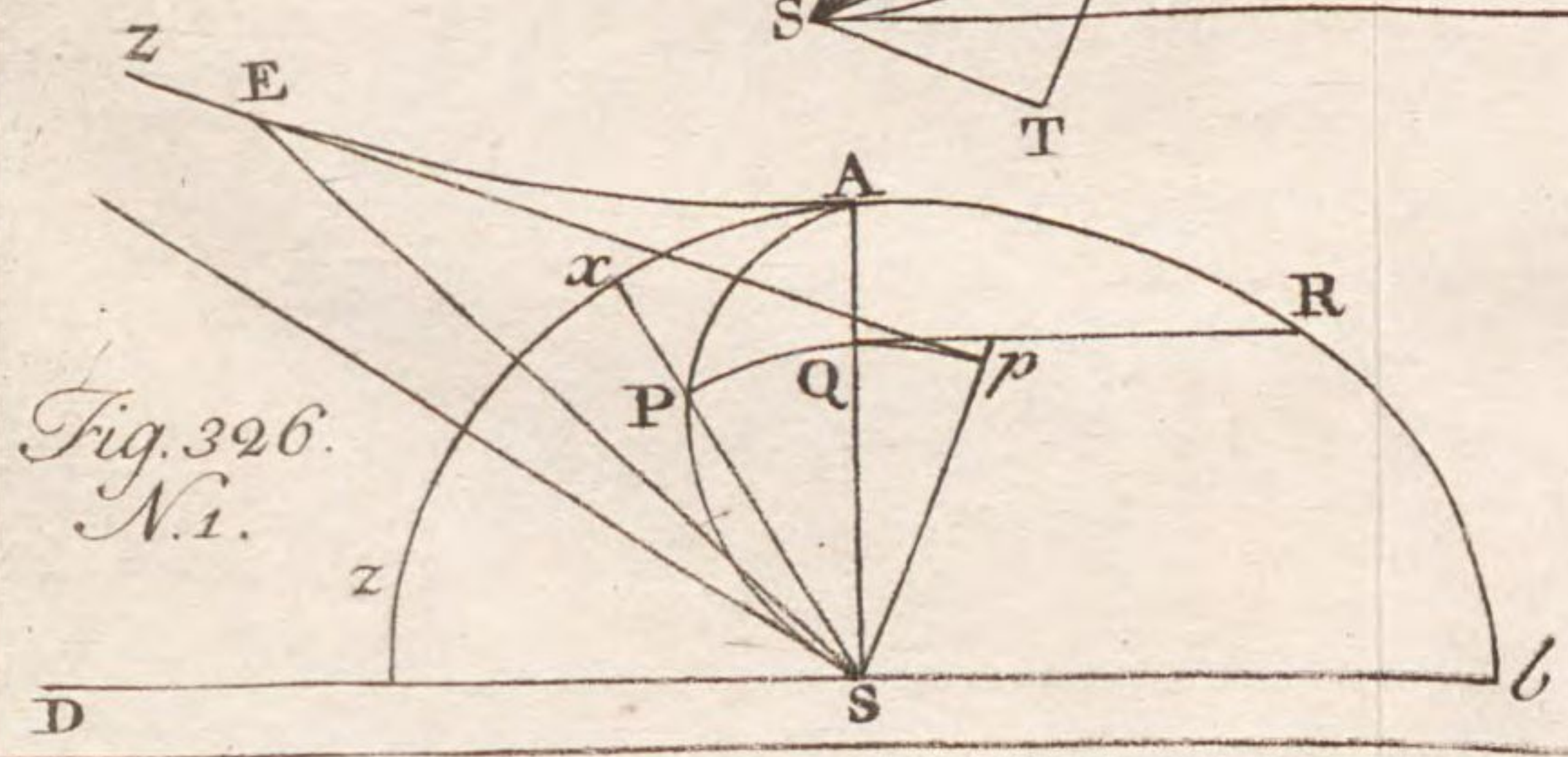
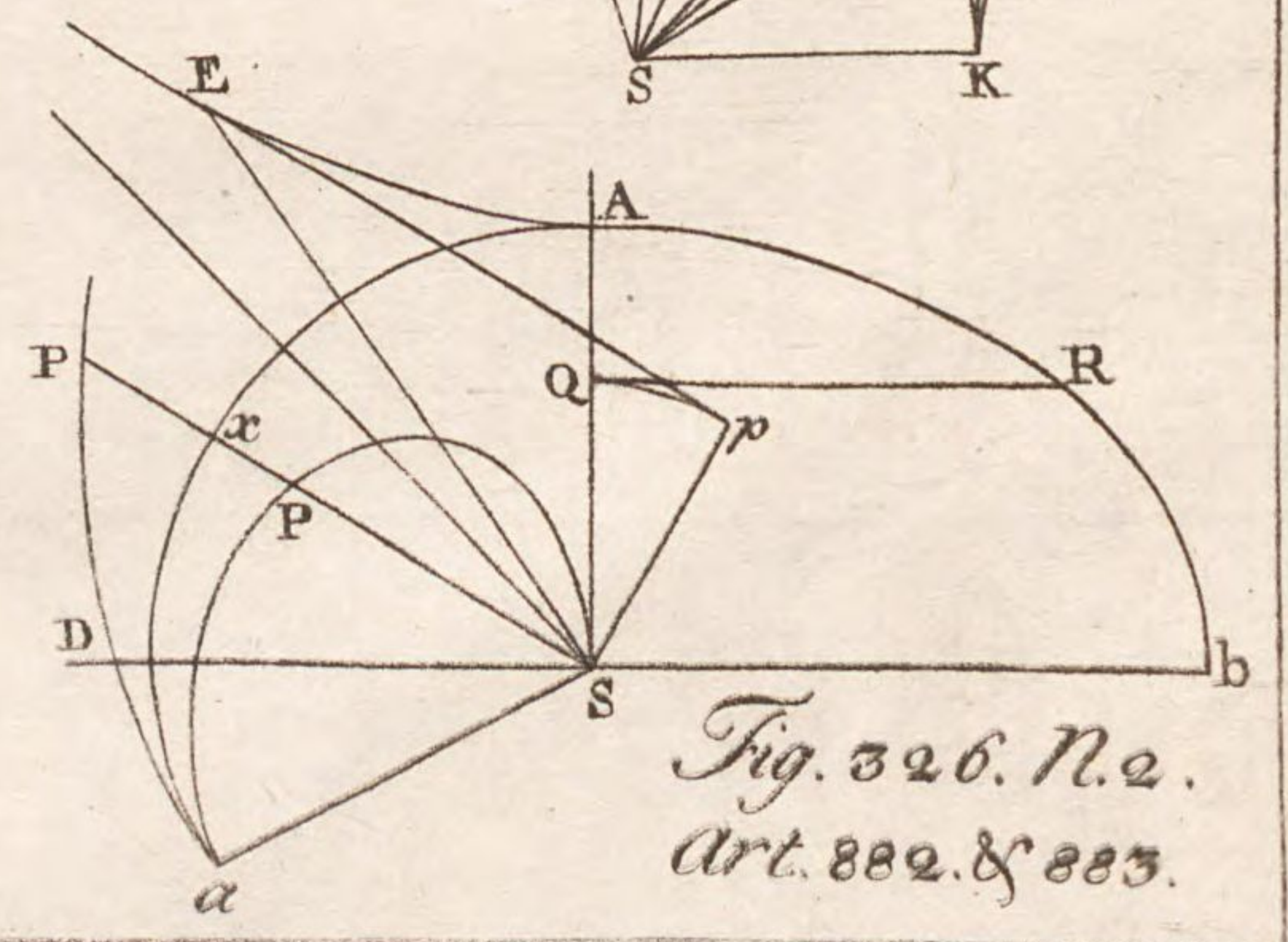


Fig. 326. N.2. Art. 882. & 883.



Small lettered words are not to be read

Fig. 1

Fig. 2

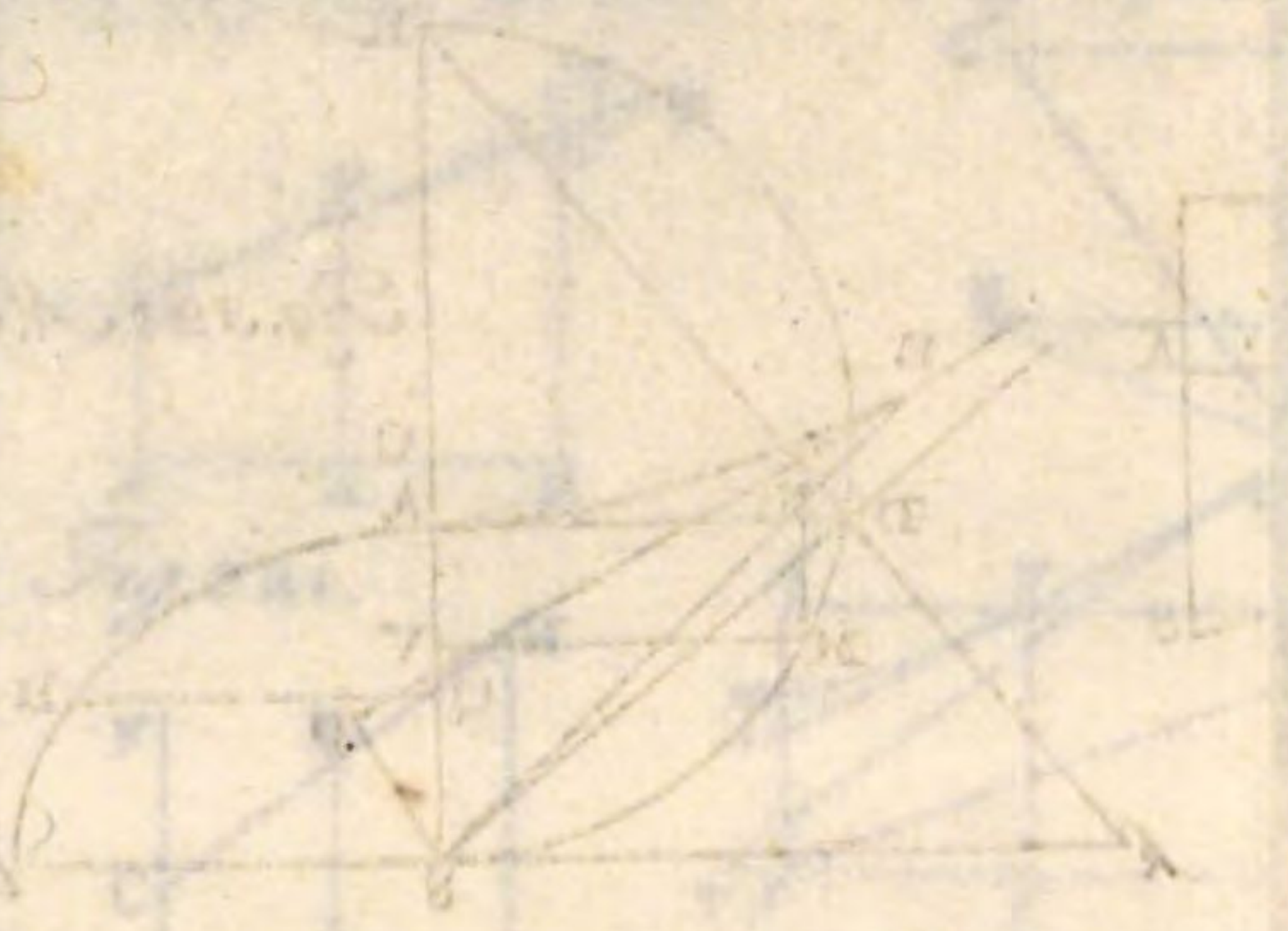
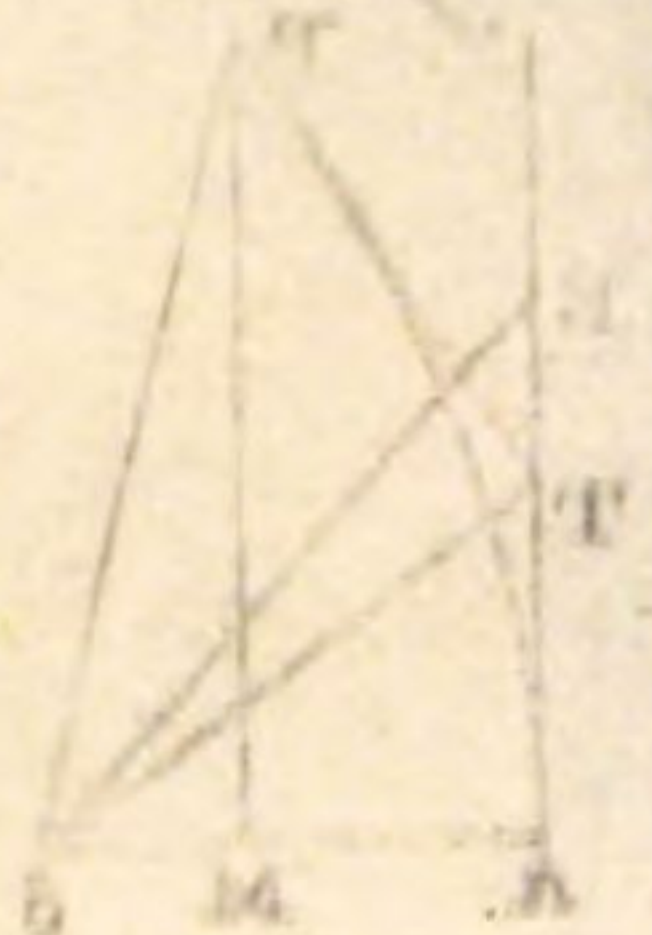


Fig. 3

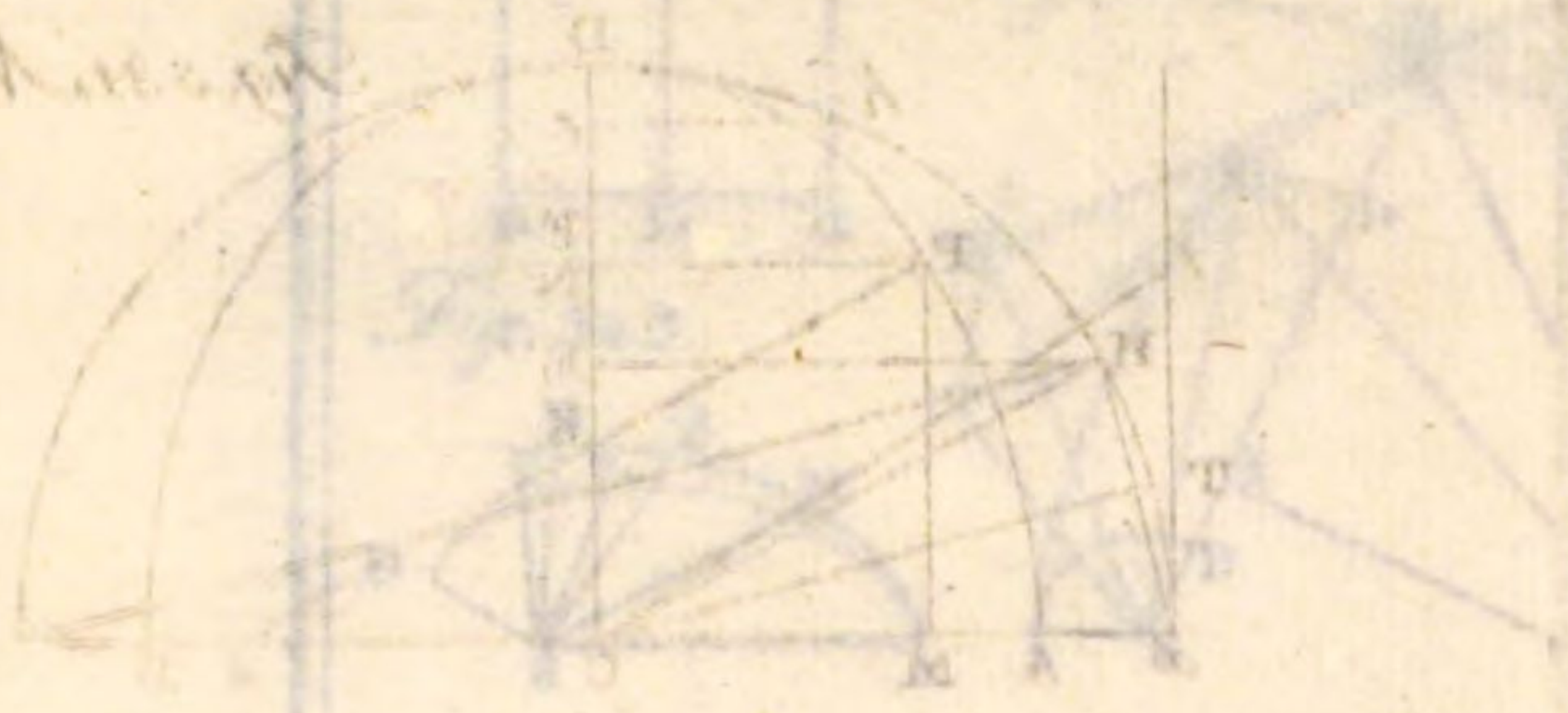


Fig. 4

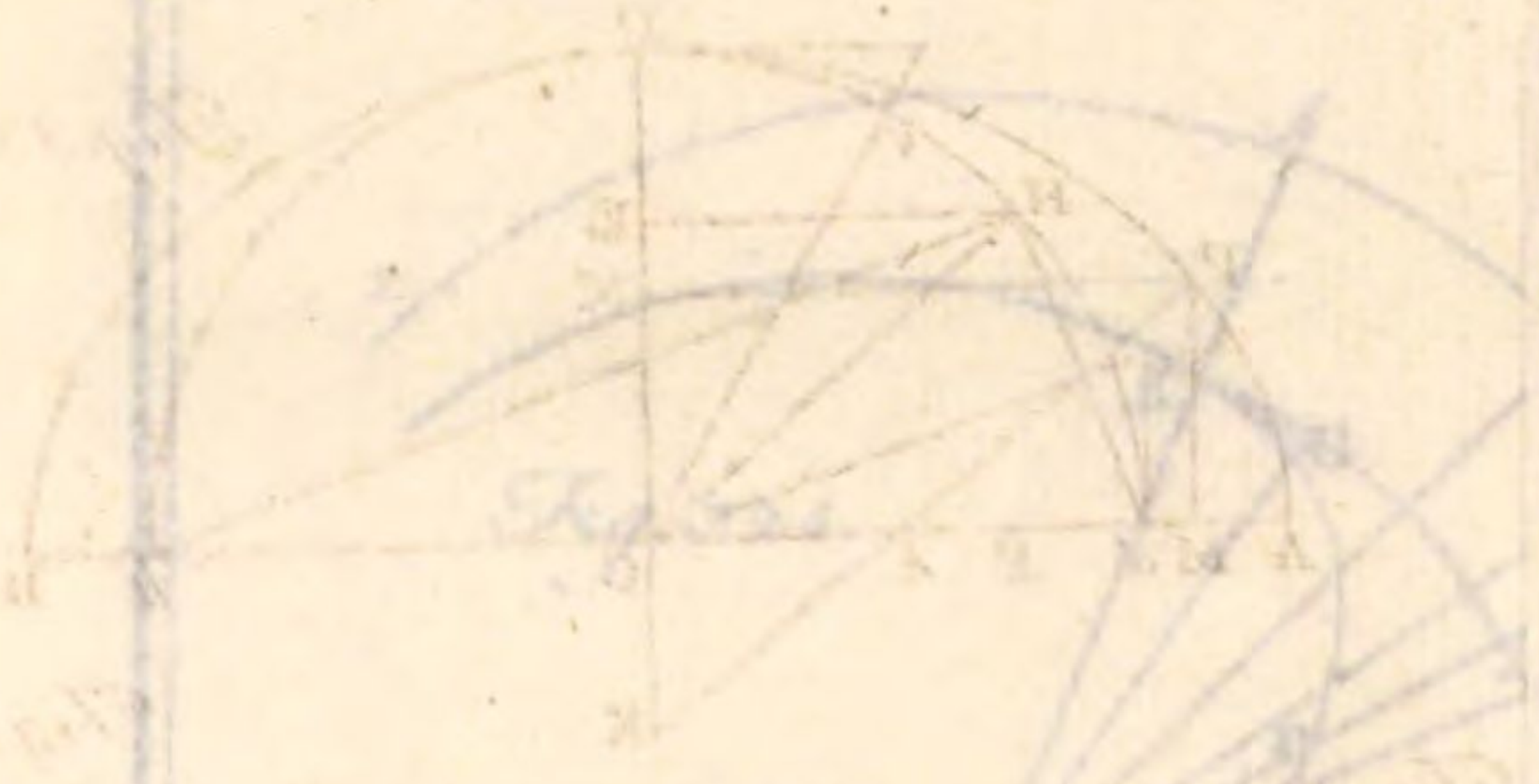
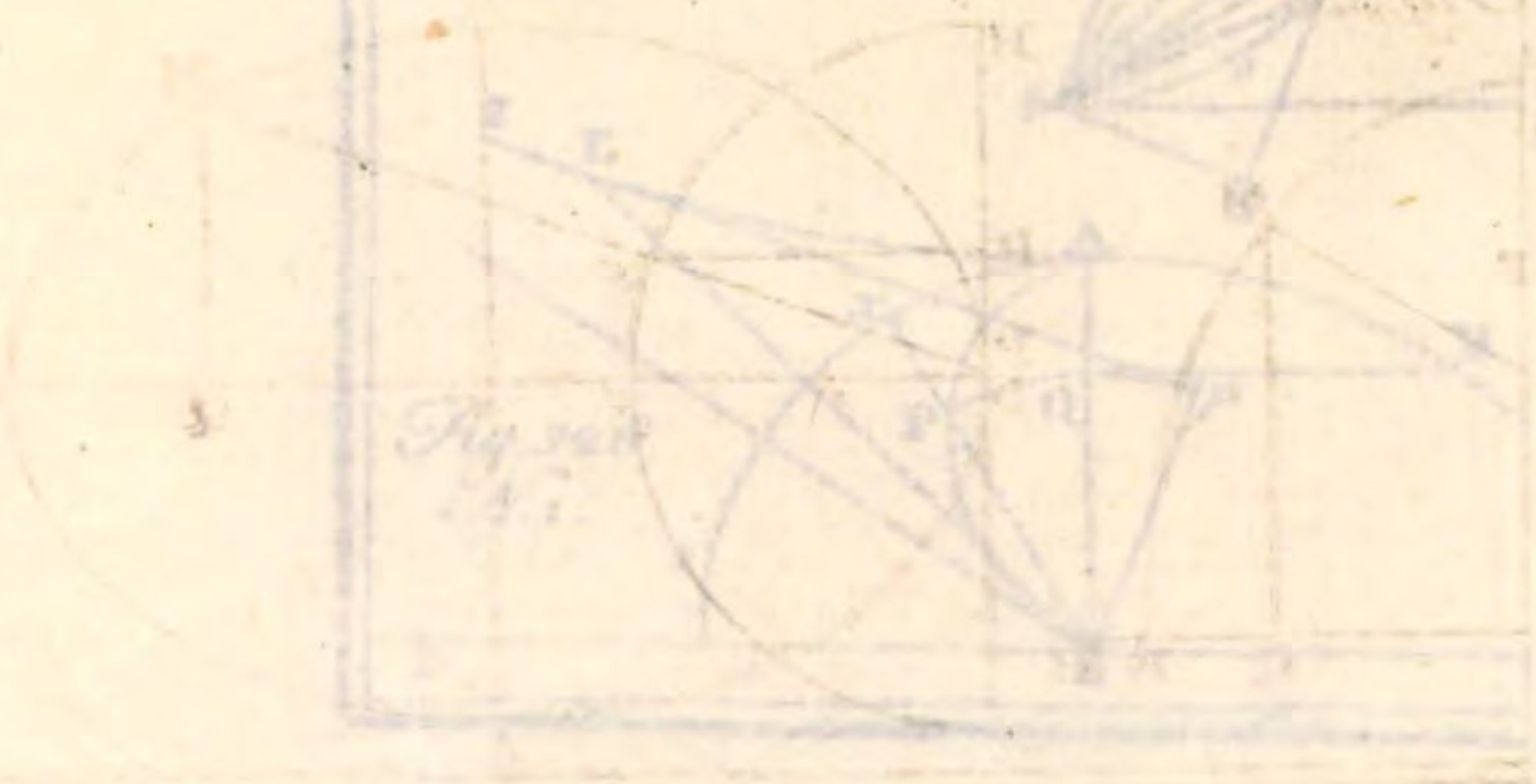


Fig. 5



ERRATA

Fig. 327.

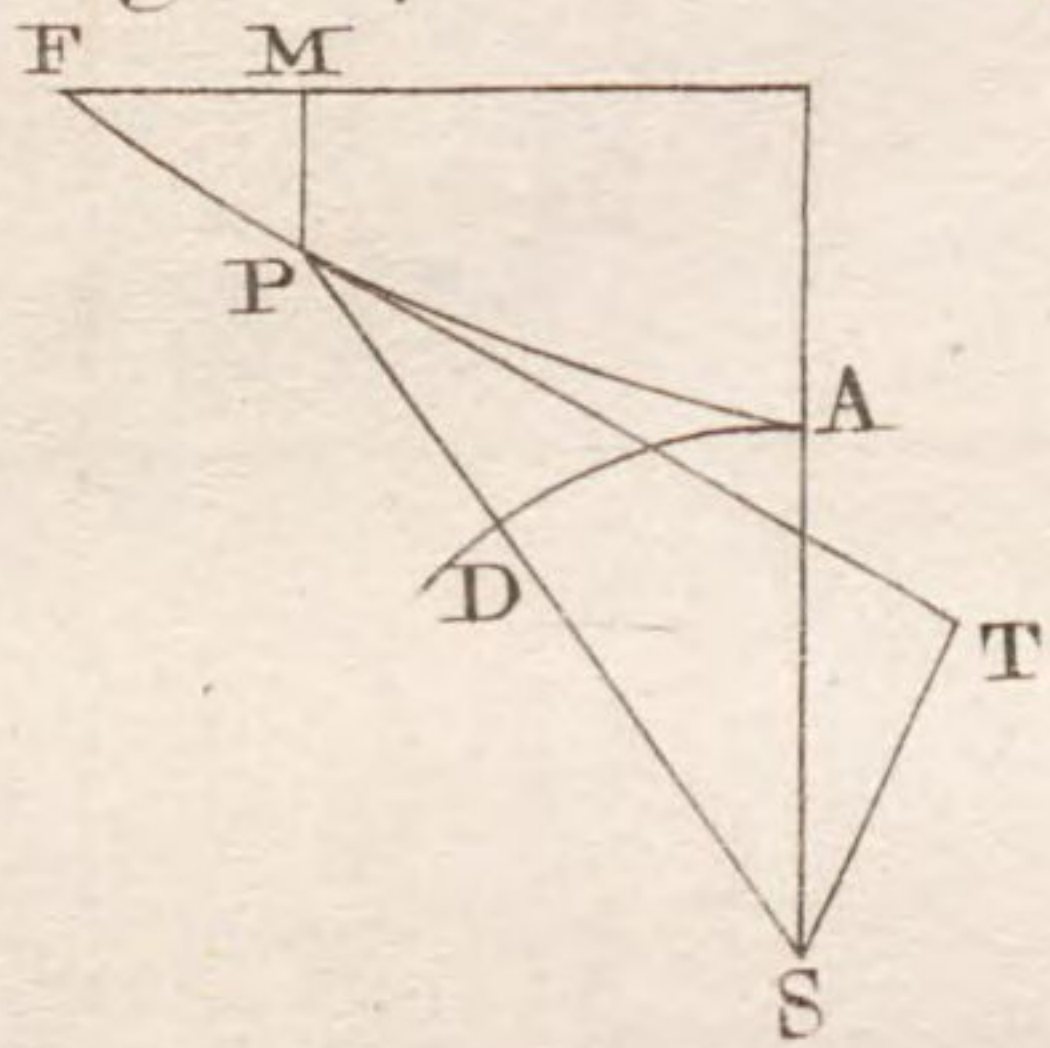


Fig. 328. N. 1.

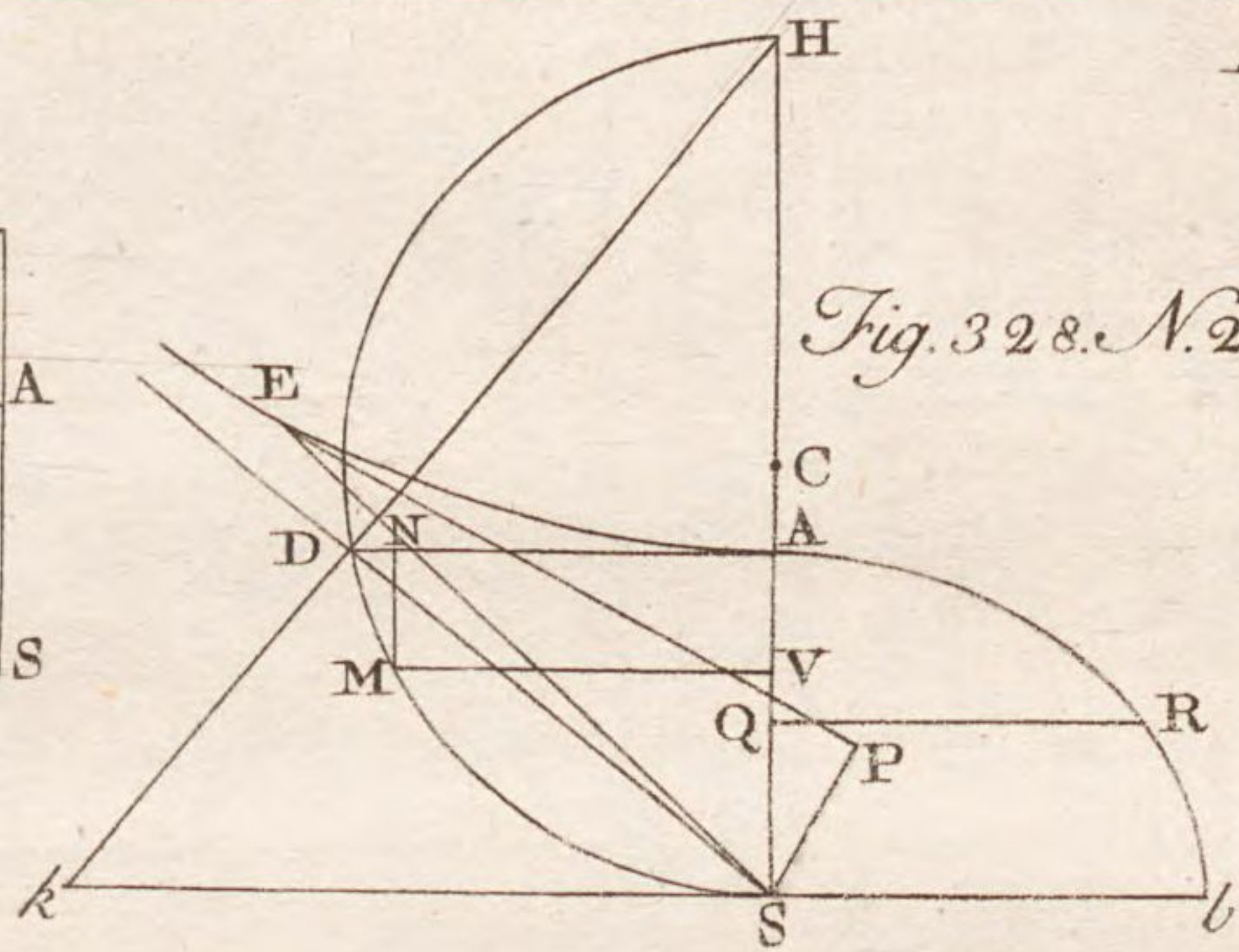
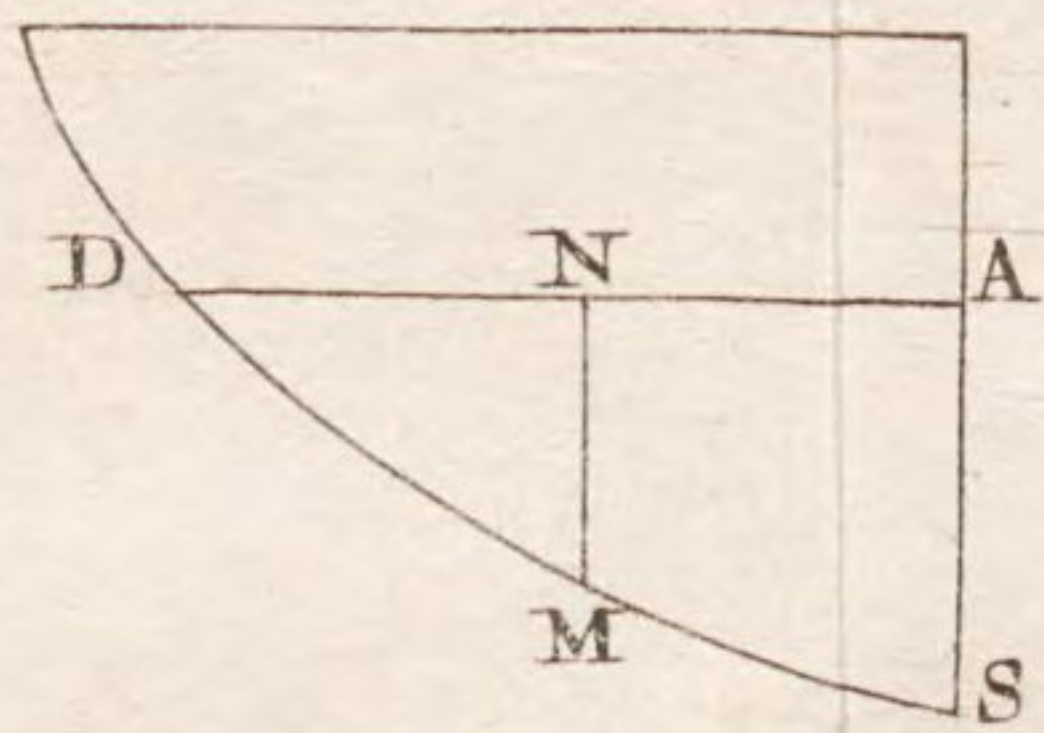


Fig. 329.

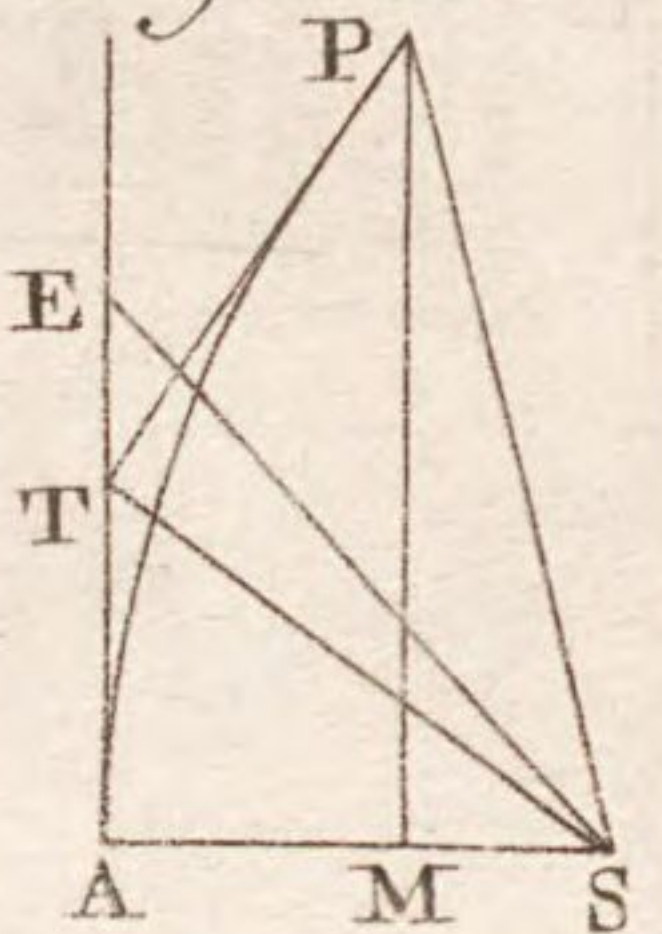


Fig. 328. N. 2.

Fig. 330.

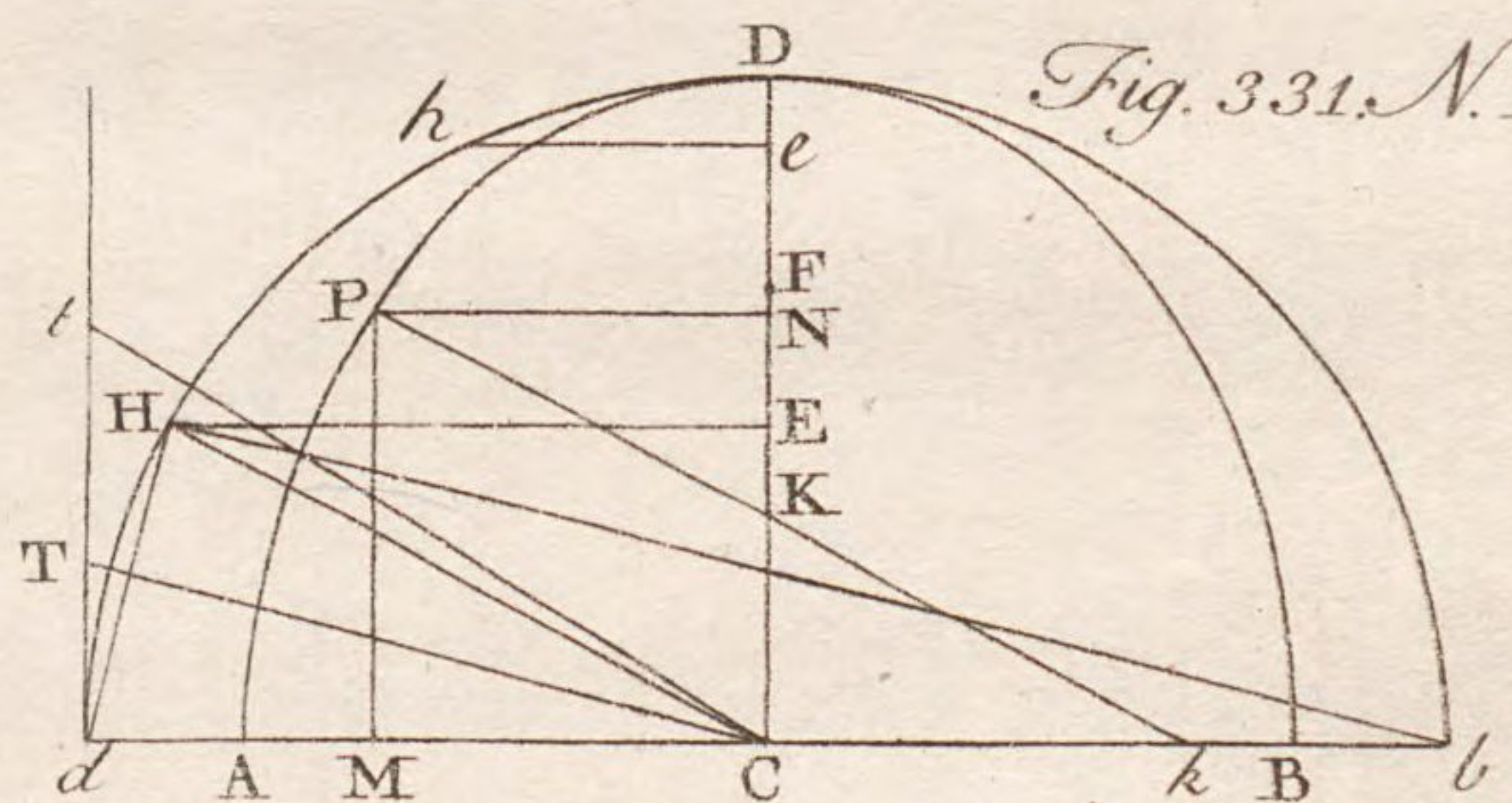
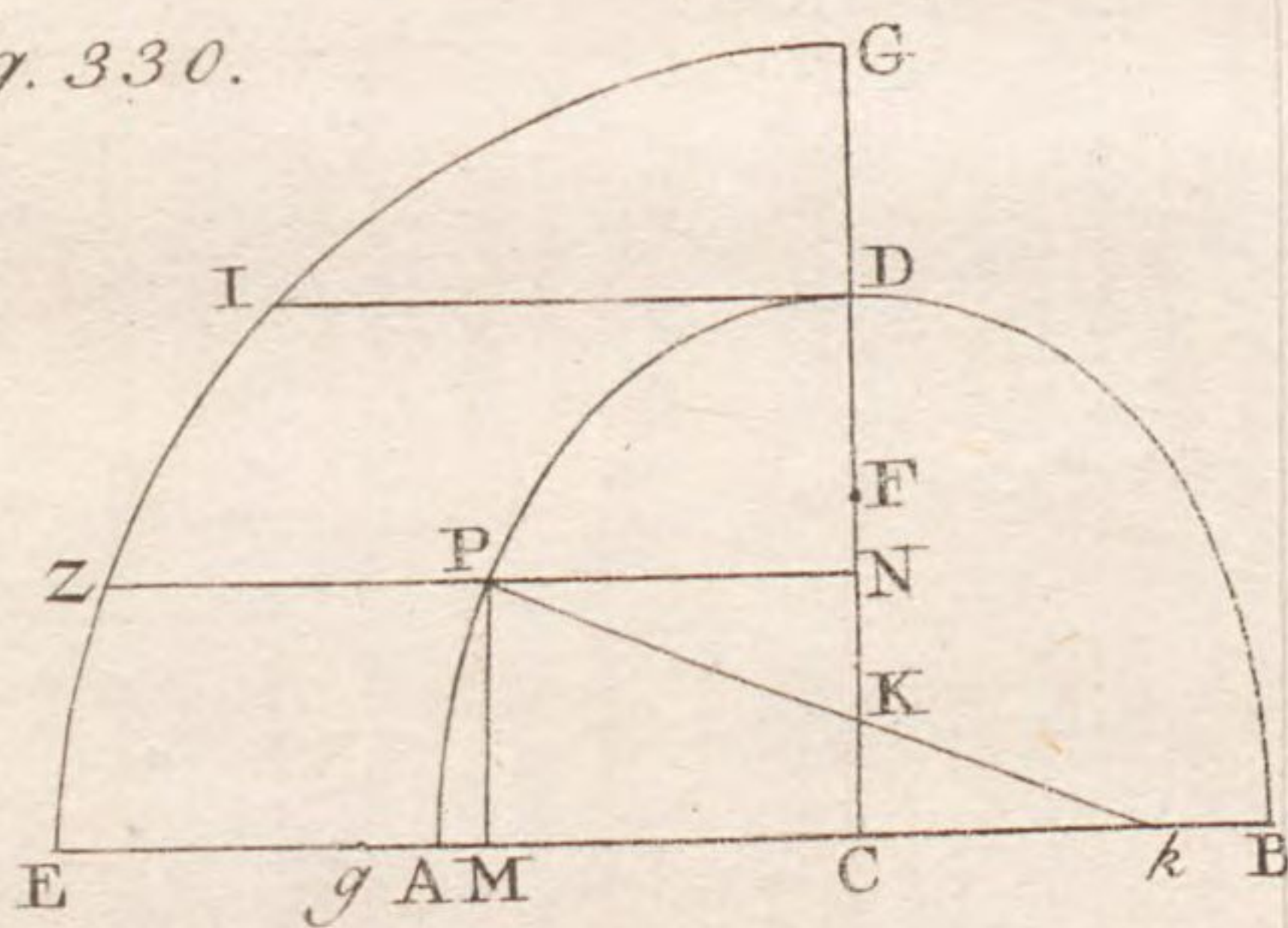


Fig. 331. N. 1.

Fig. 332. N. 1.

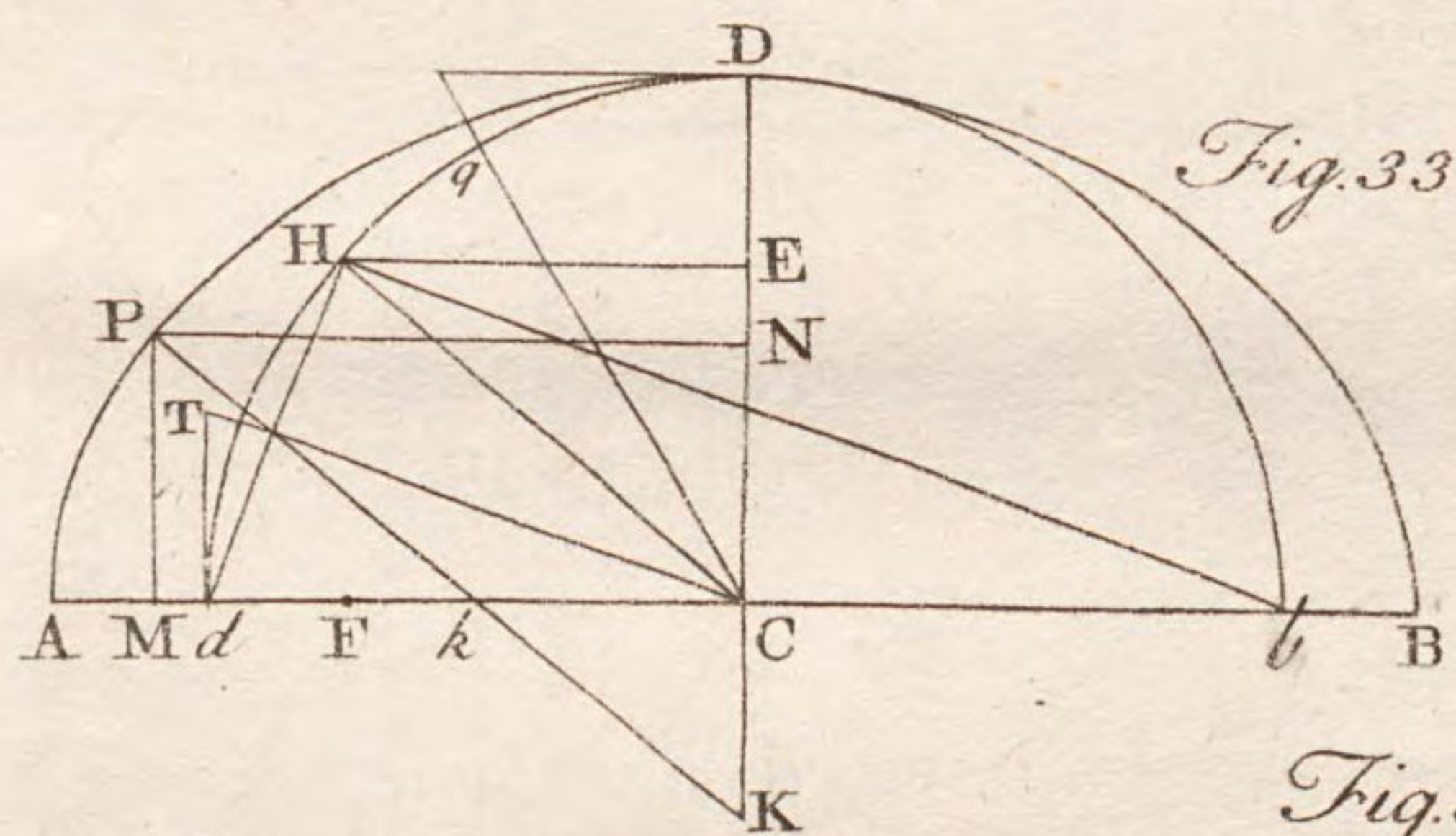
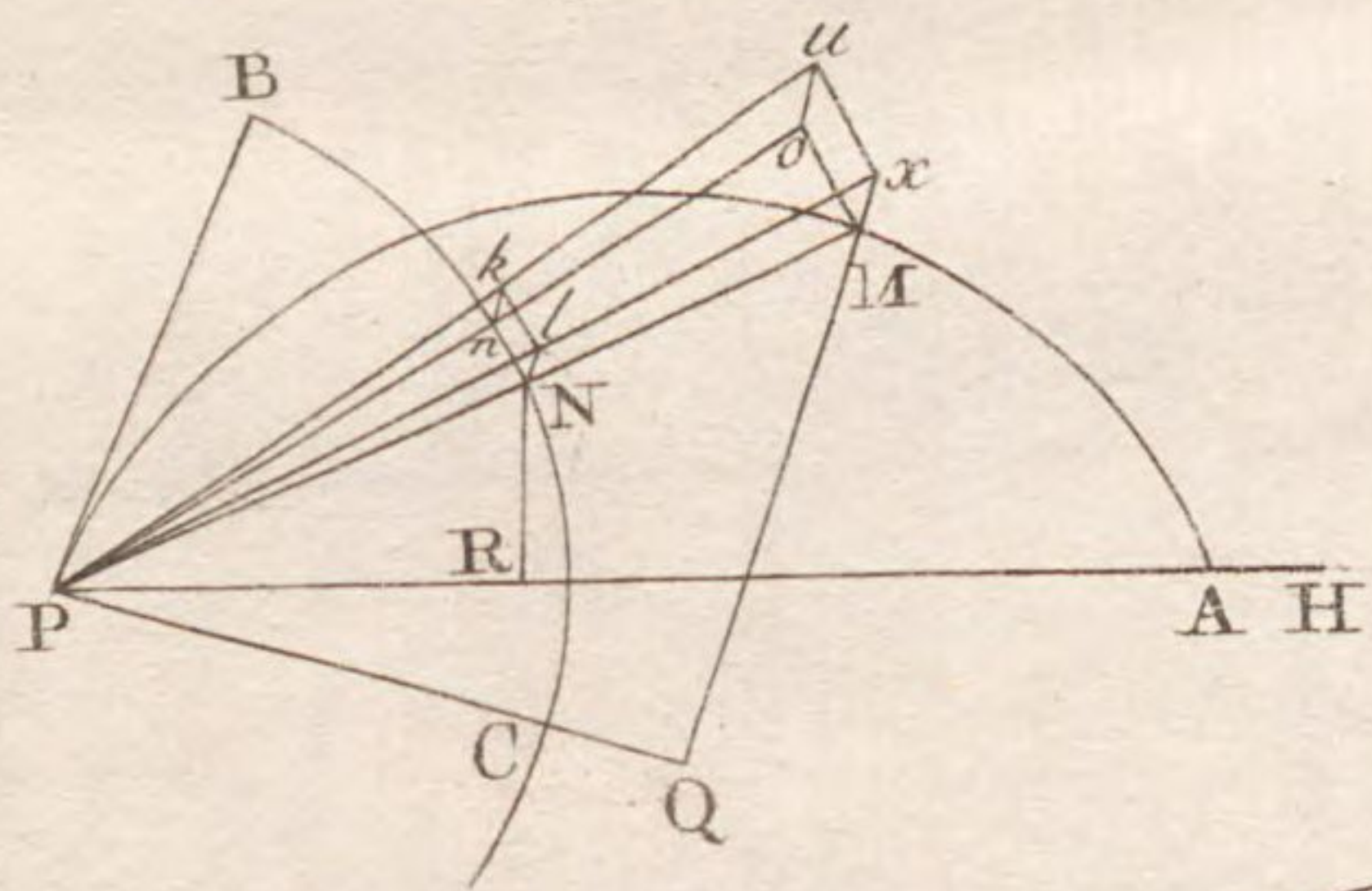


Fig. 331. N. 2.

Fig. 333.

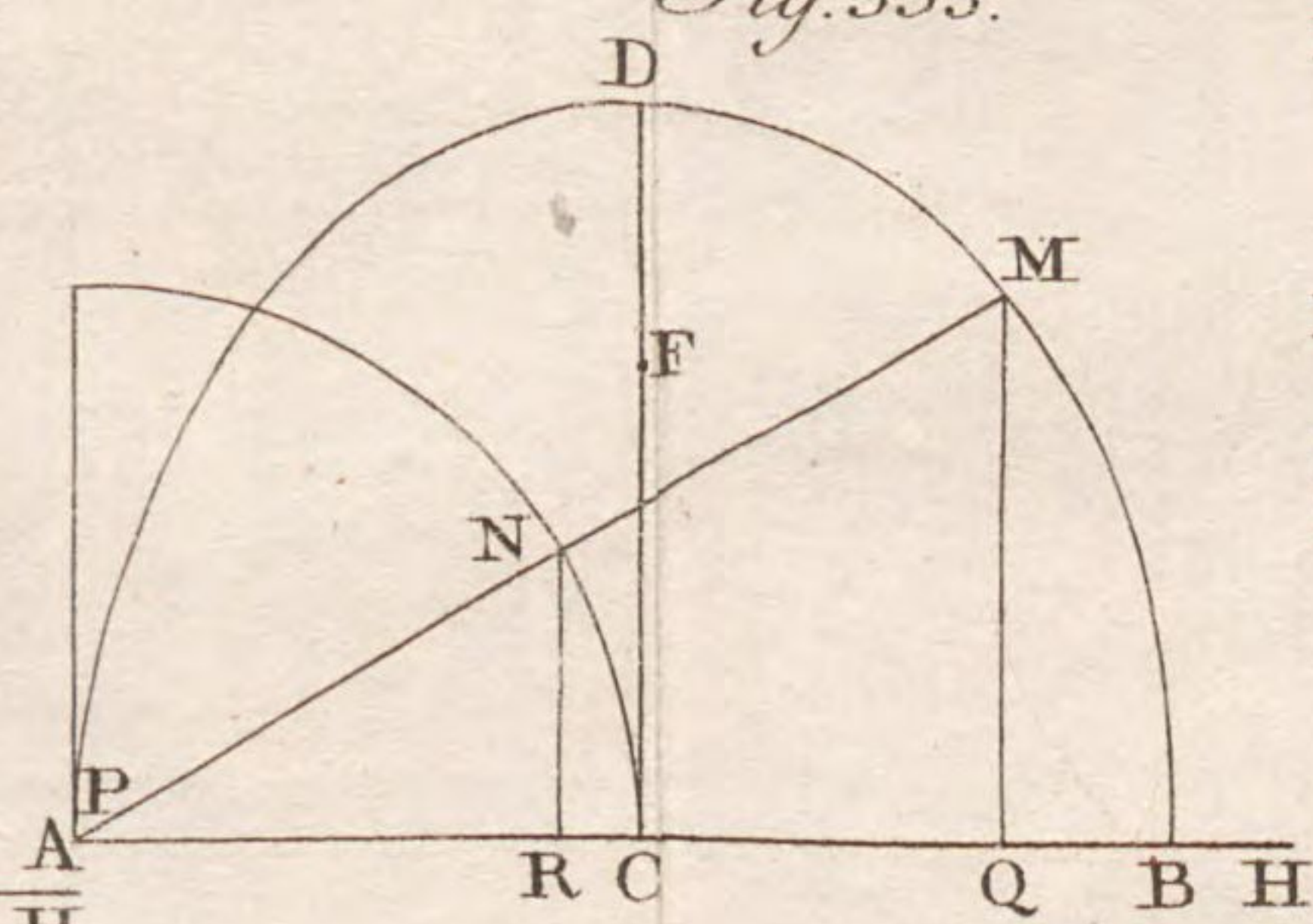


Fig. 332. N. 2.

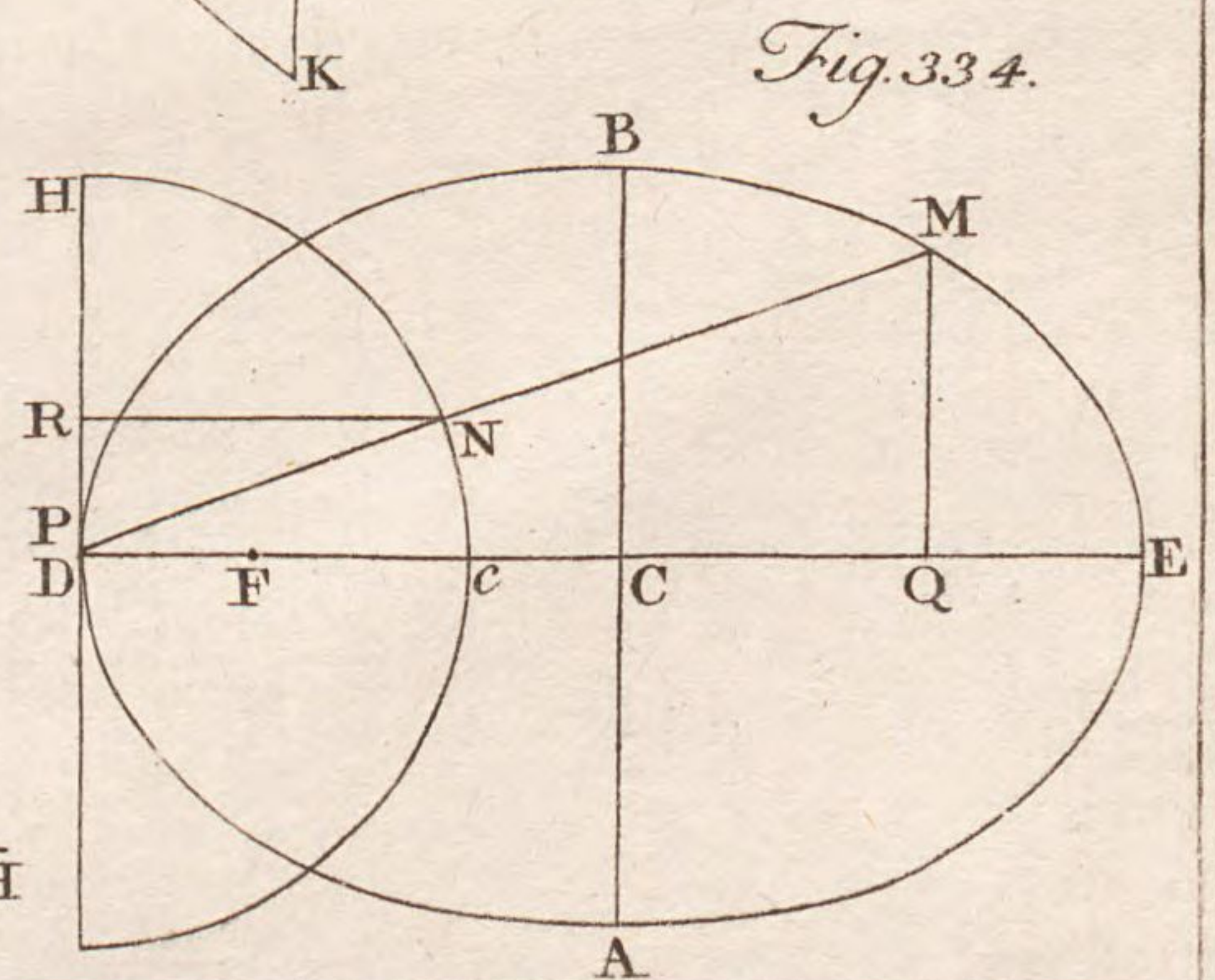
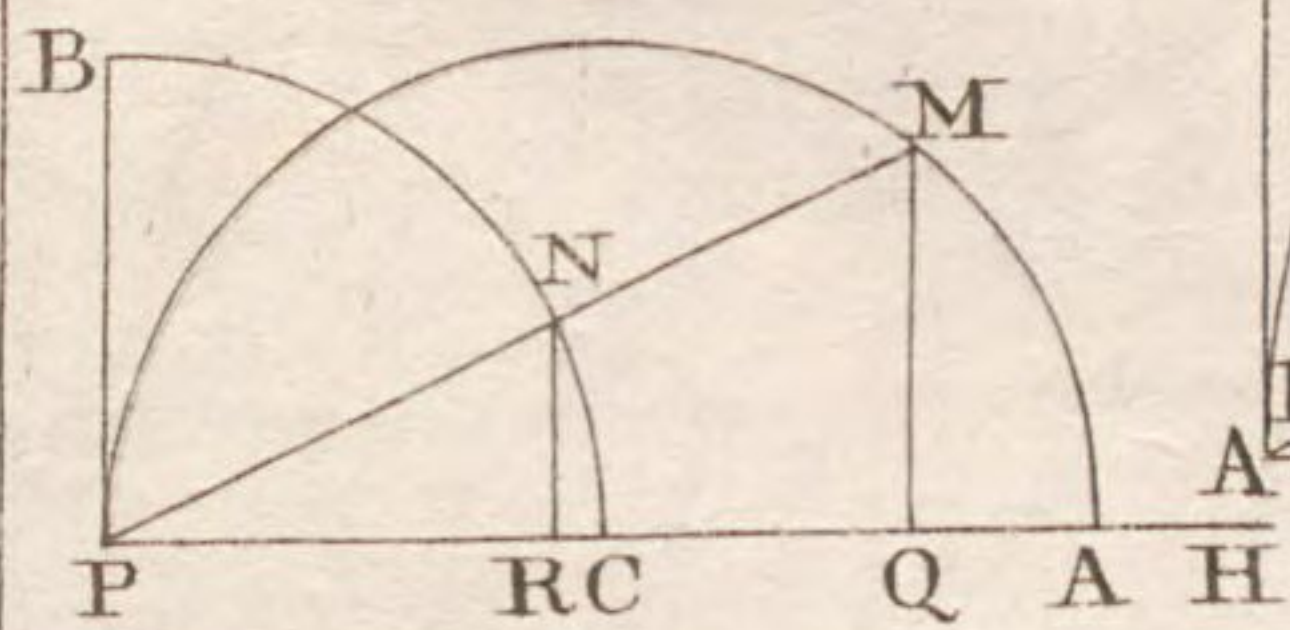


Fig. 334.

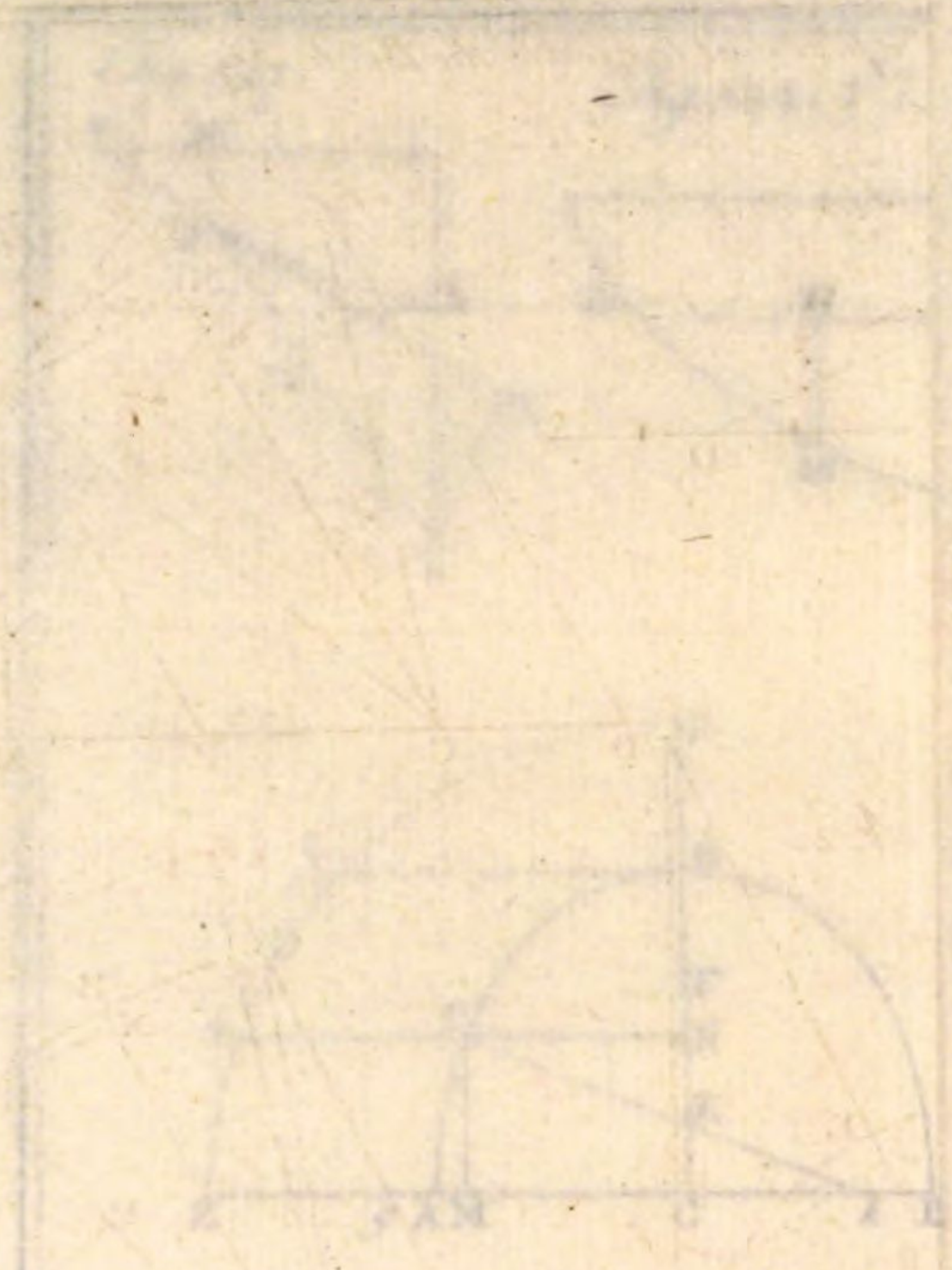
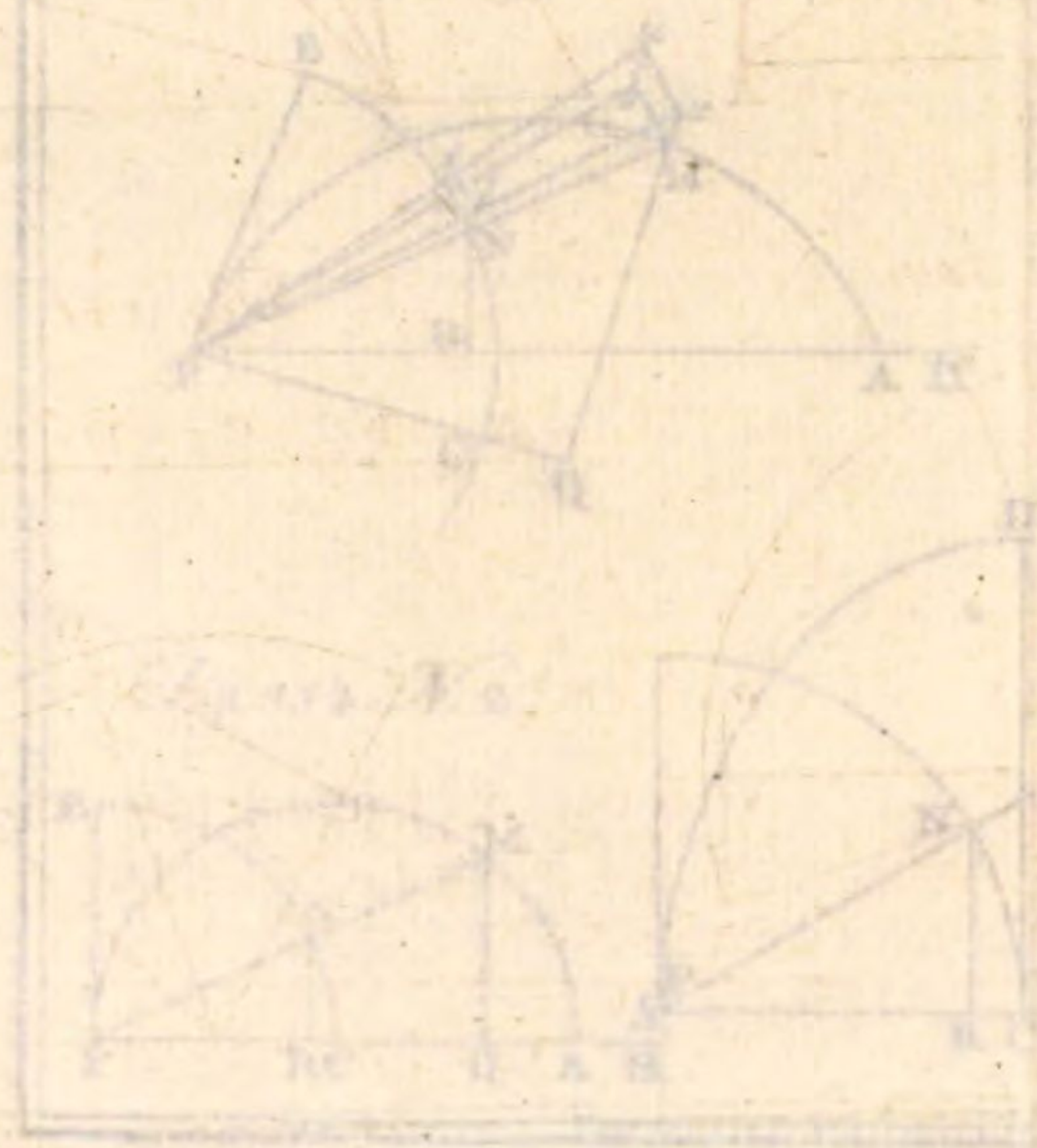


Fig. 302. M.C.



To be plac'd at the end of the Book

Pl. XXXIX.

Fig. 335.

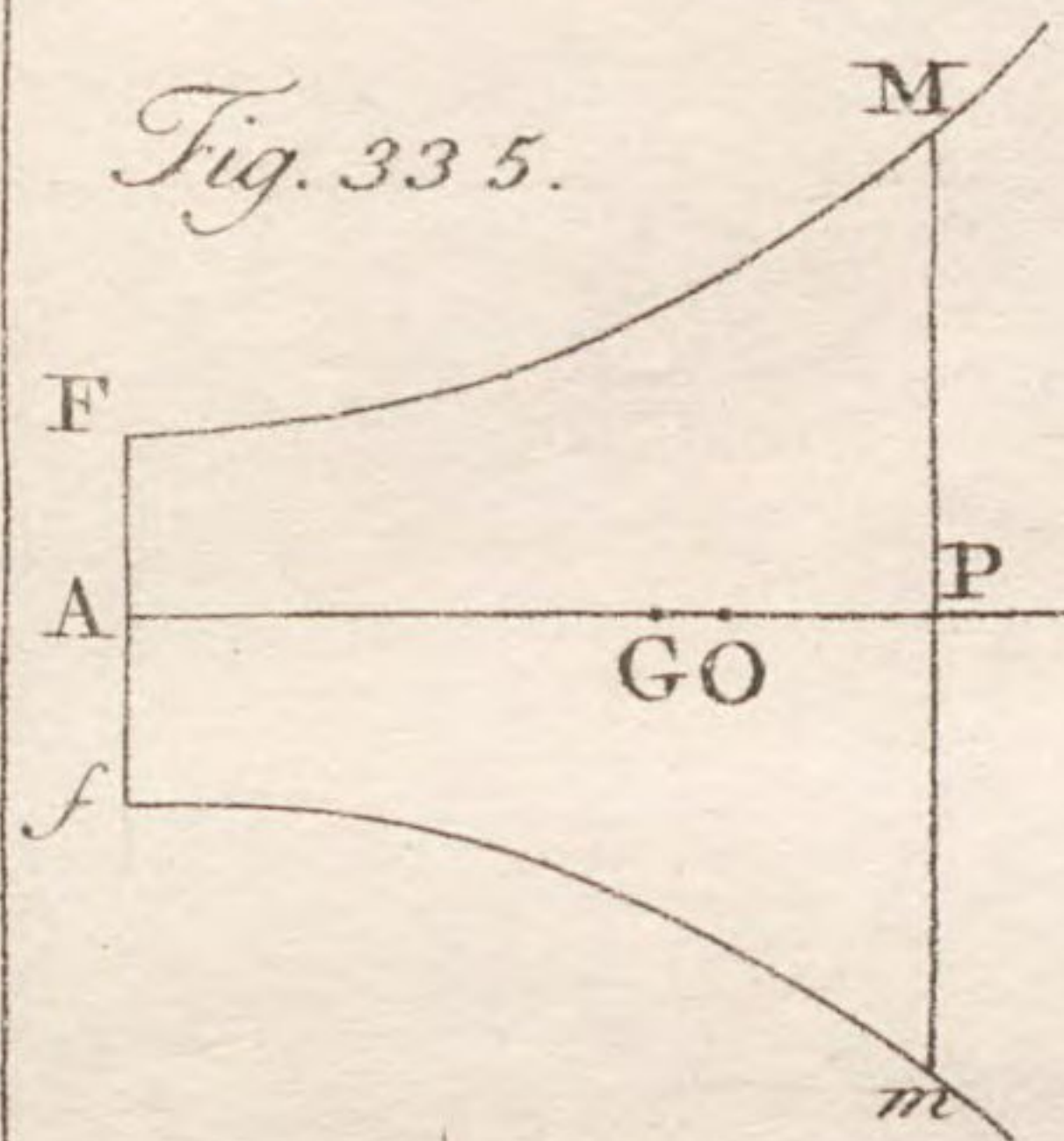


Fig. 336.

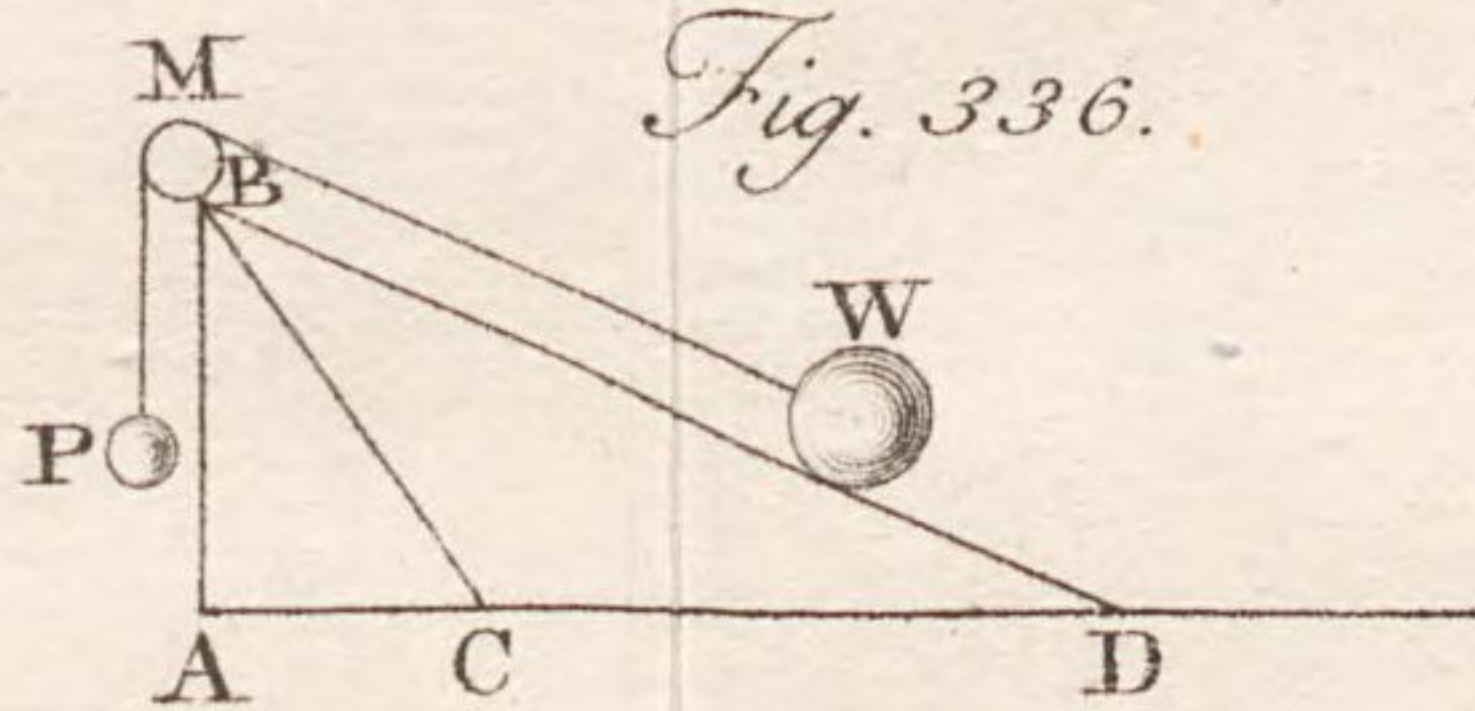


Fig. 337. N. 1.

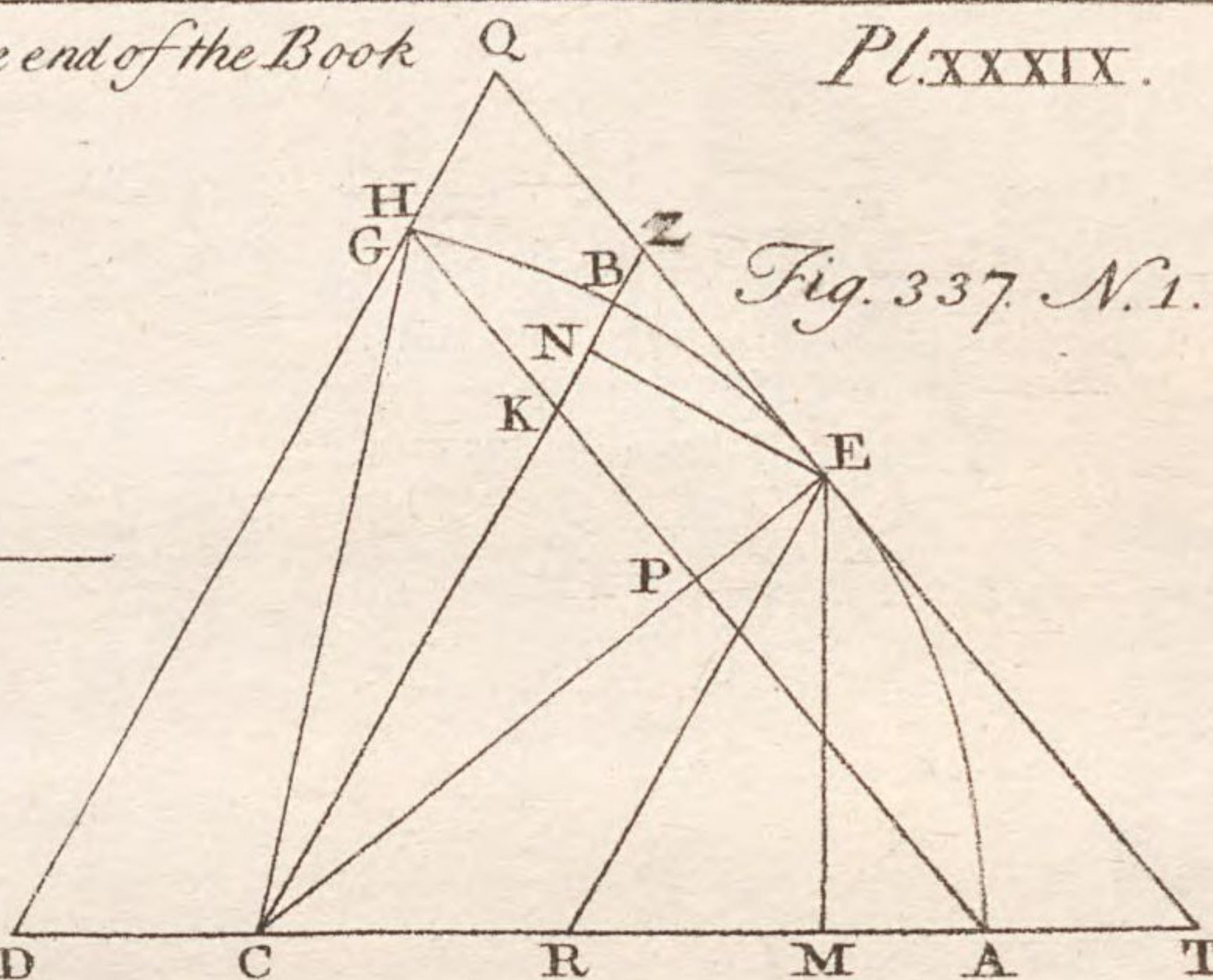


Fig. 338. N. 1.

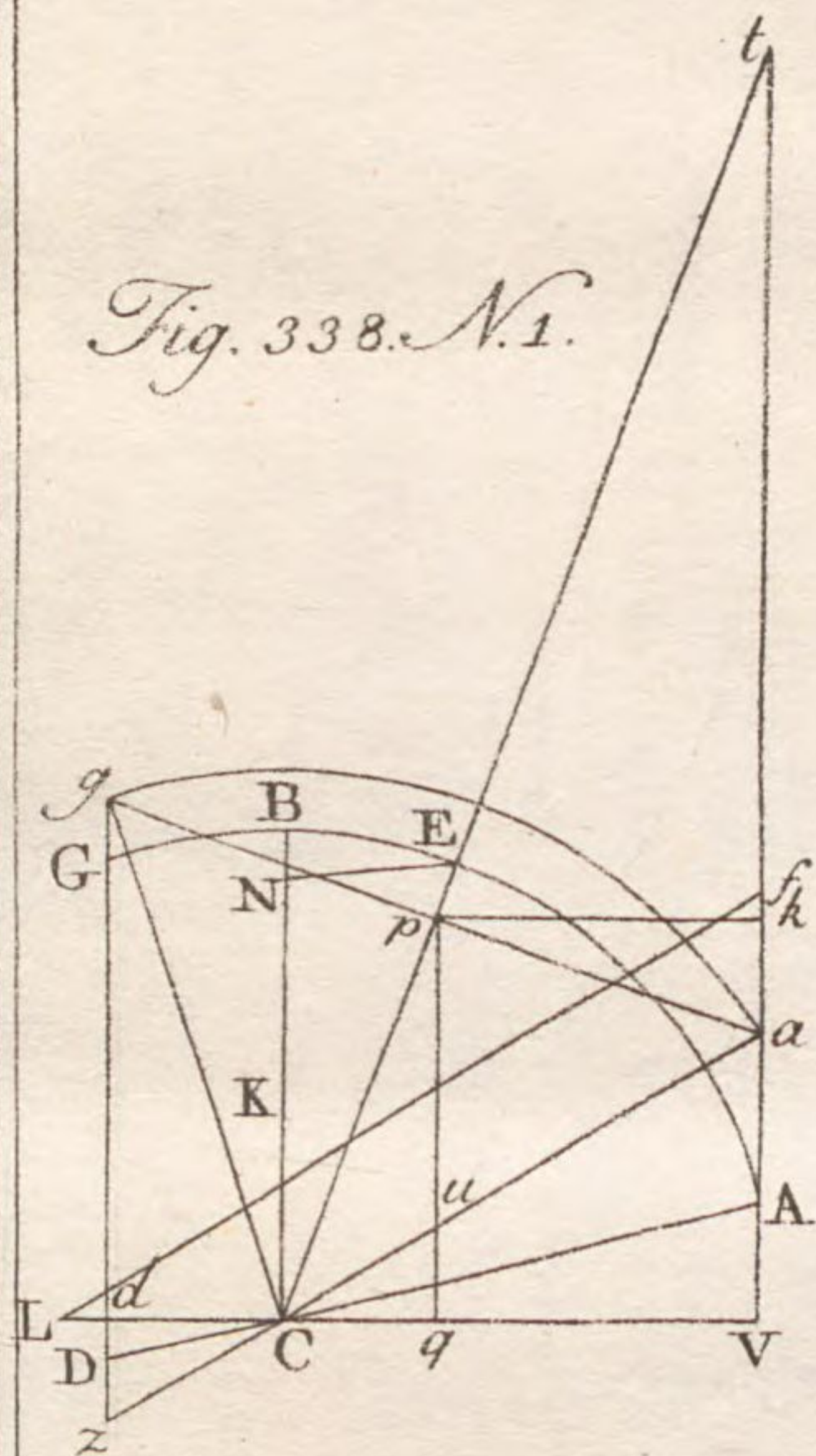


Fig. 338. N. 2.

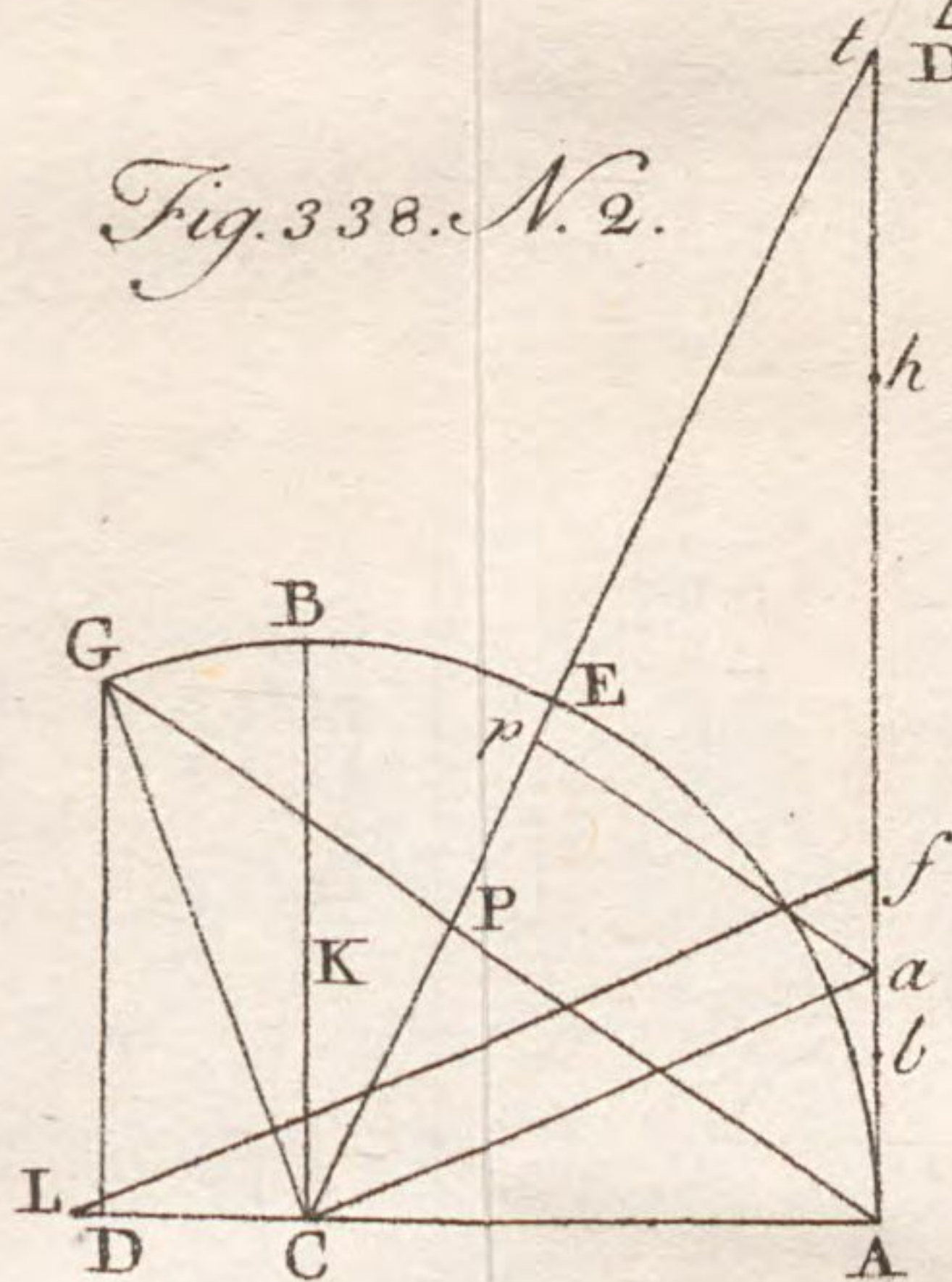


Fig. 337. N. 2.

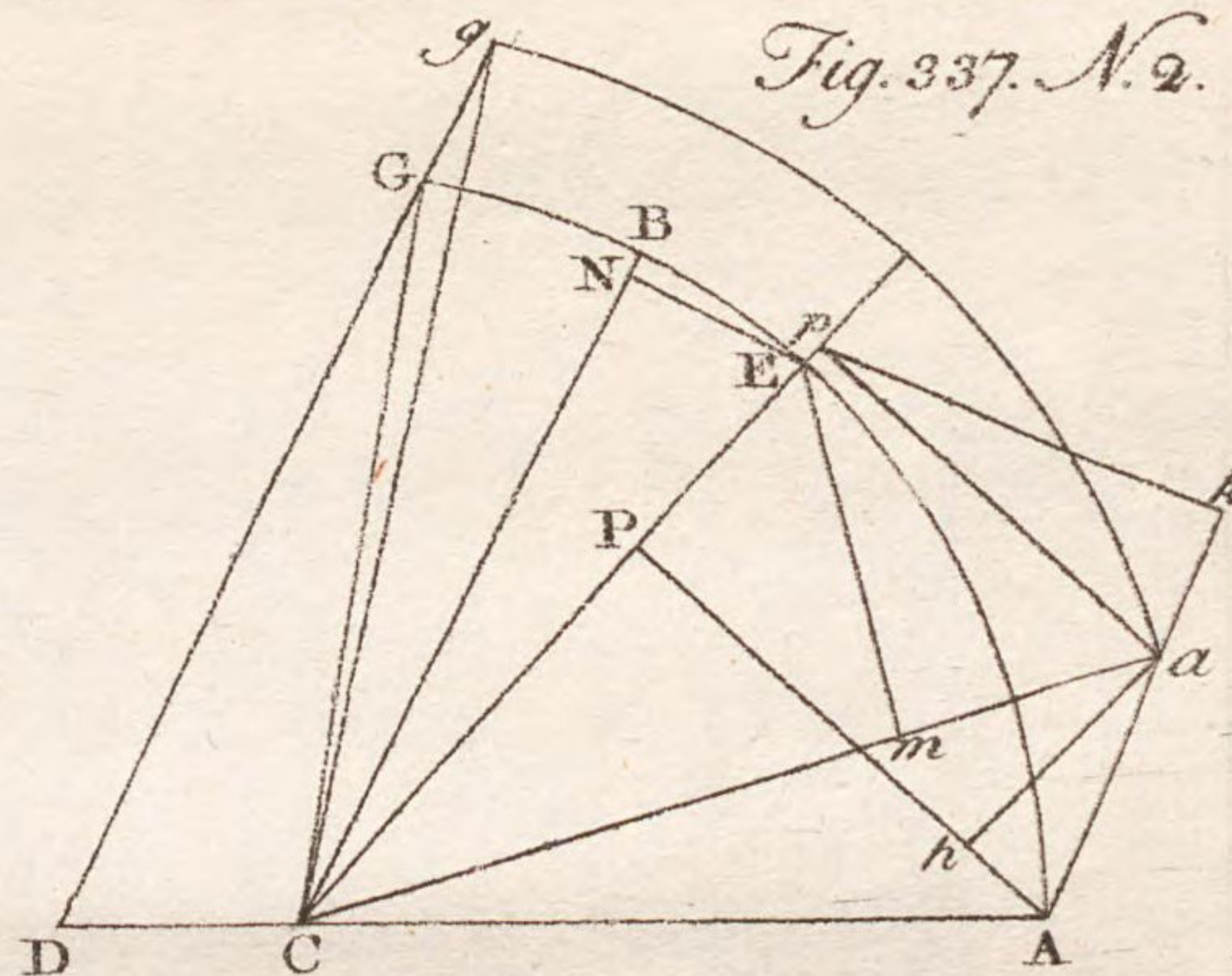


Fig. 340.

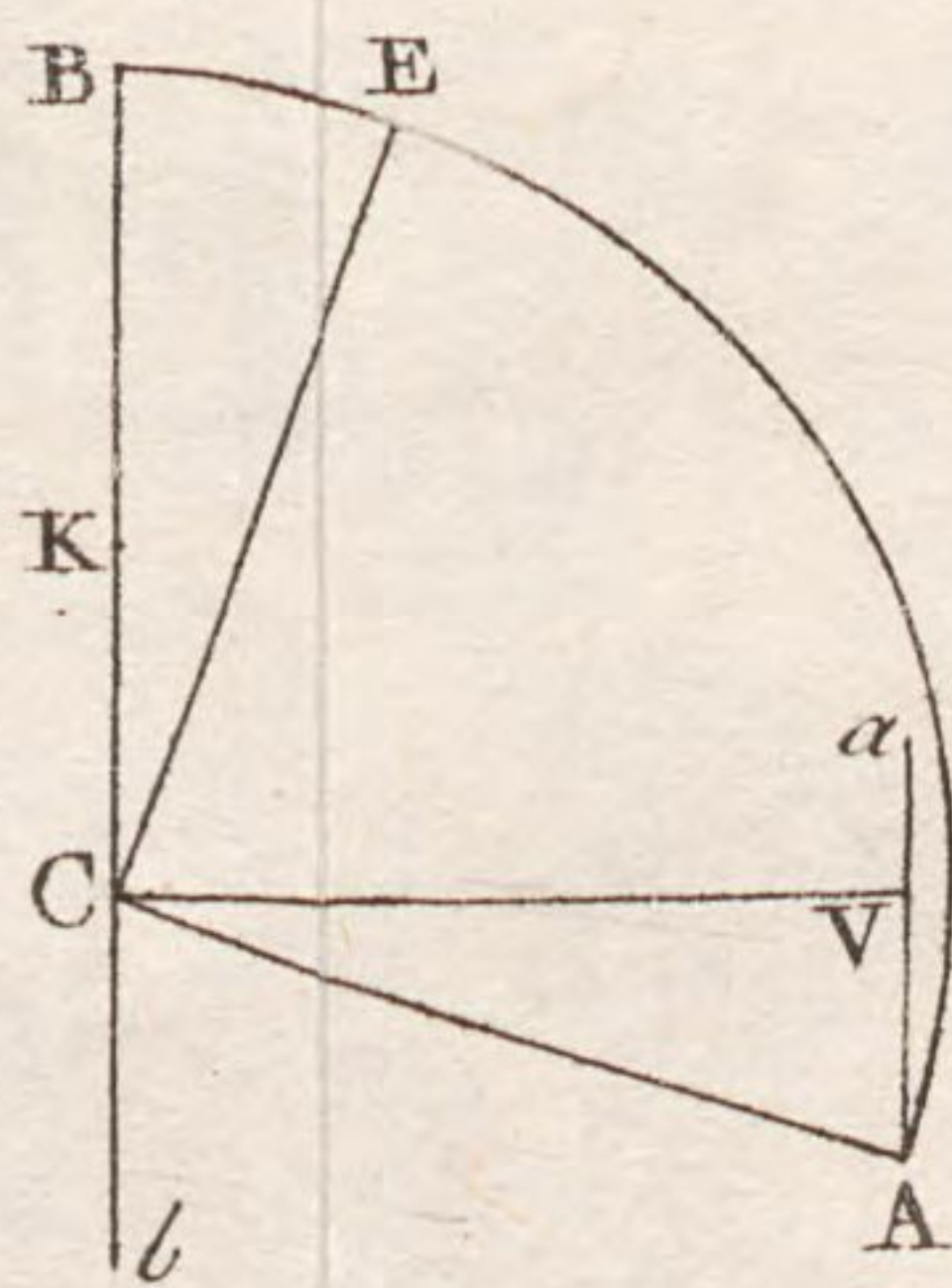


Fig. 341.

A B C c D E G

Fig. 339.

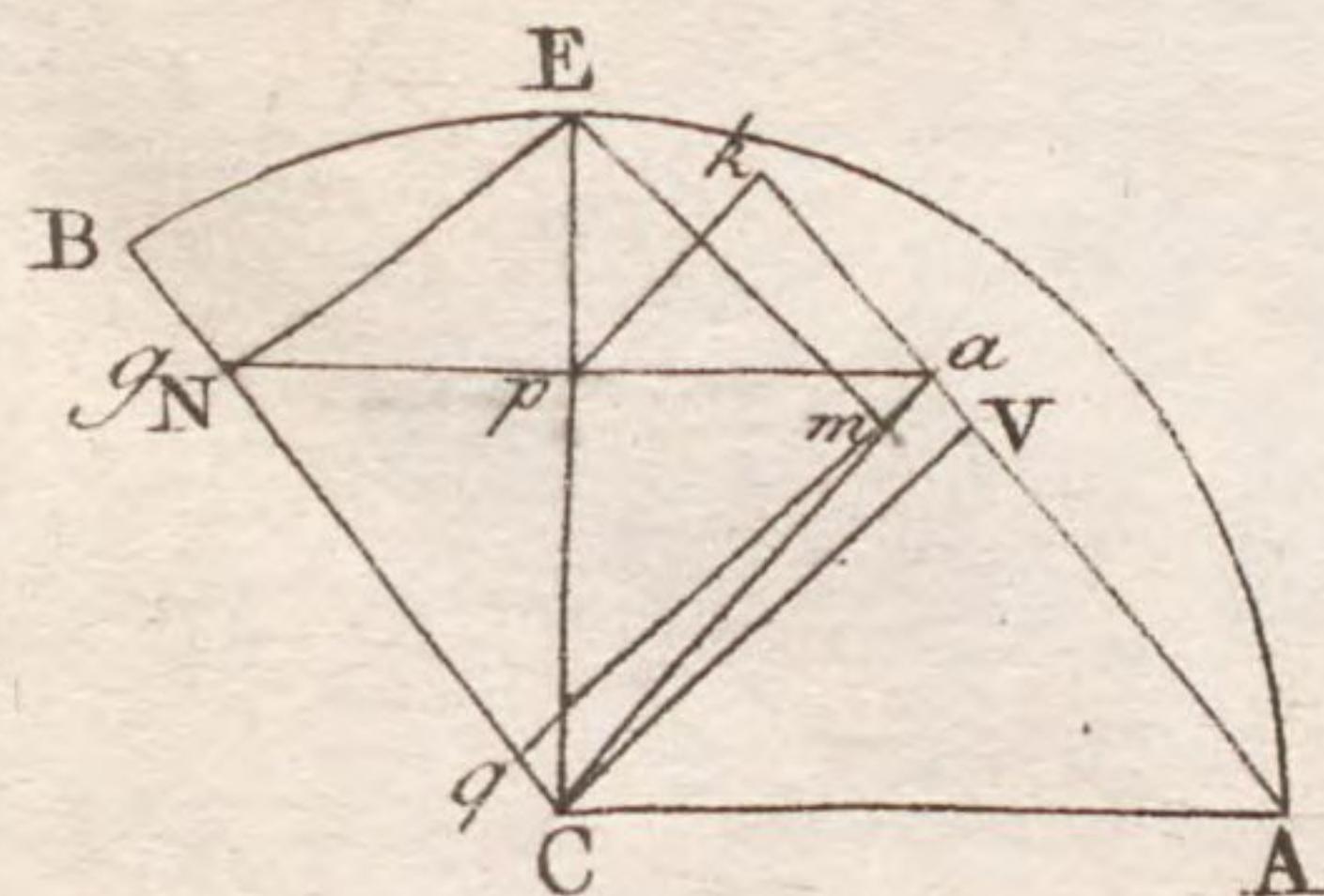
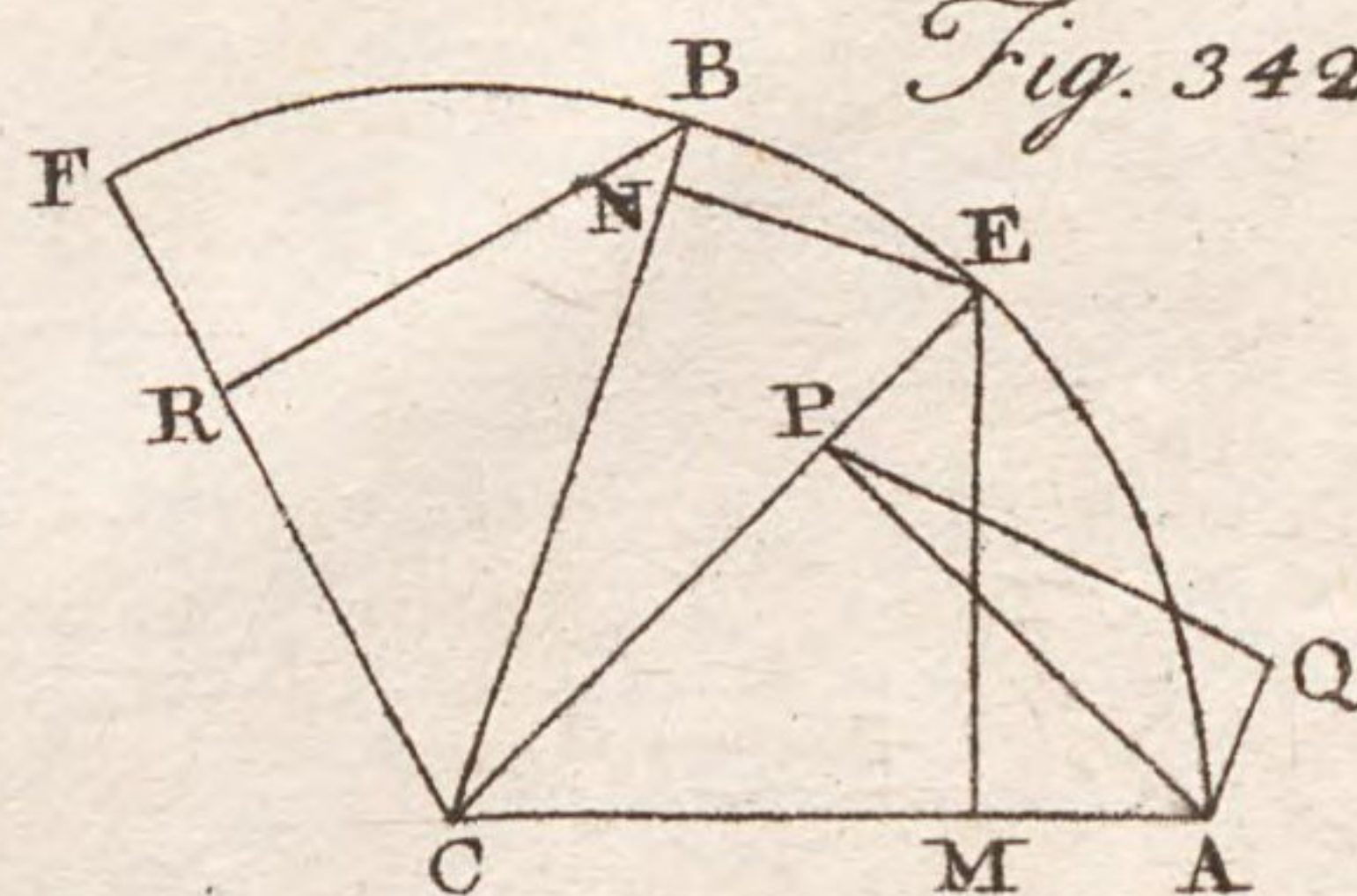


Fig. 342.



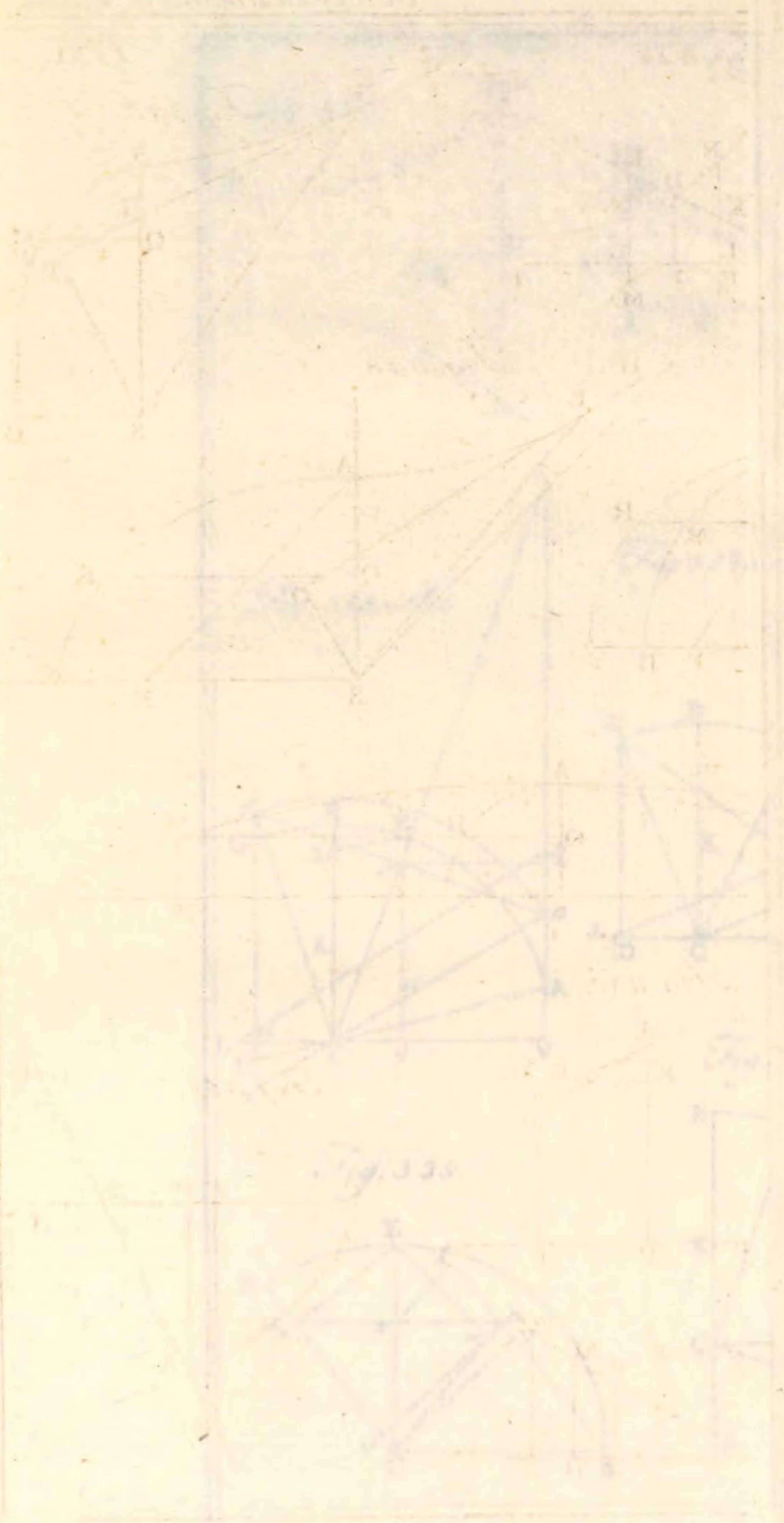


Fig. 330

Fig. 331

Fig. 343.

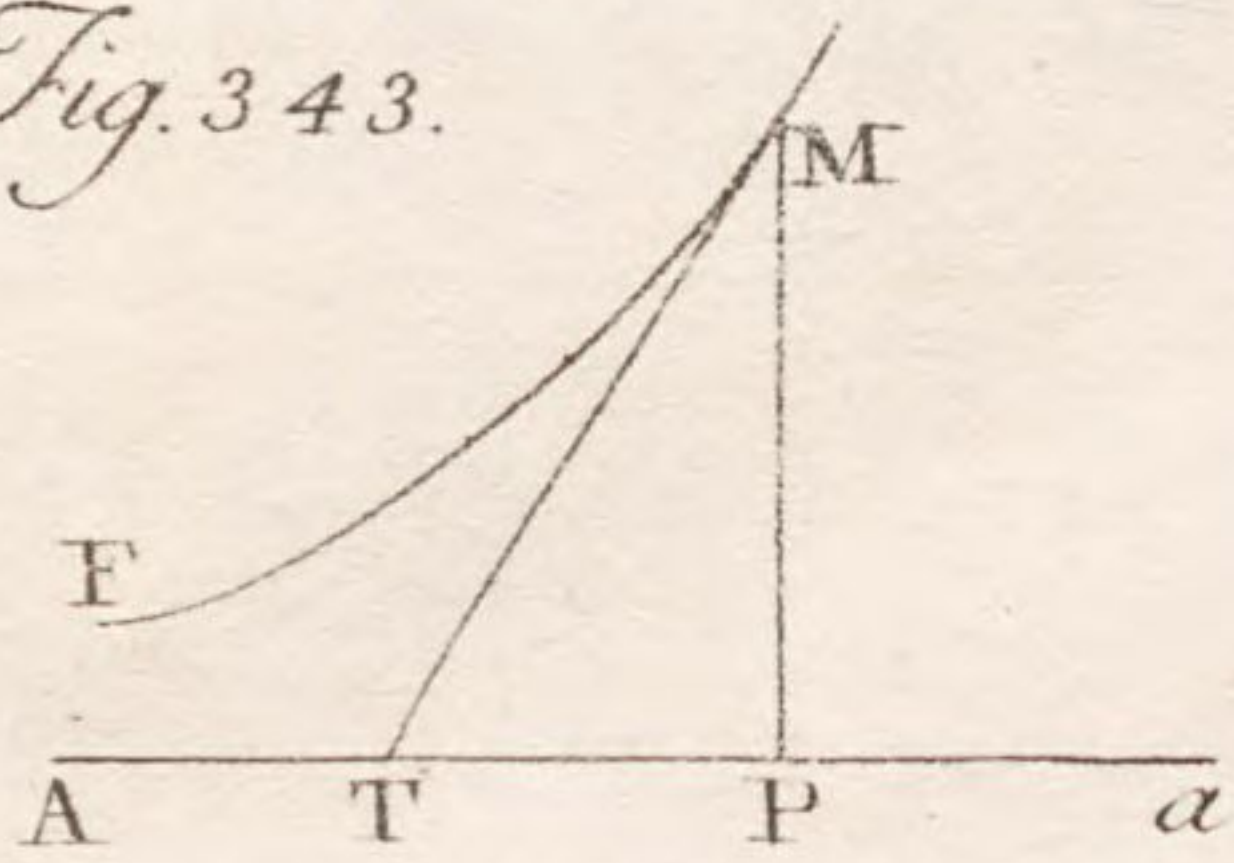


Fig. 344.

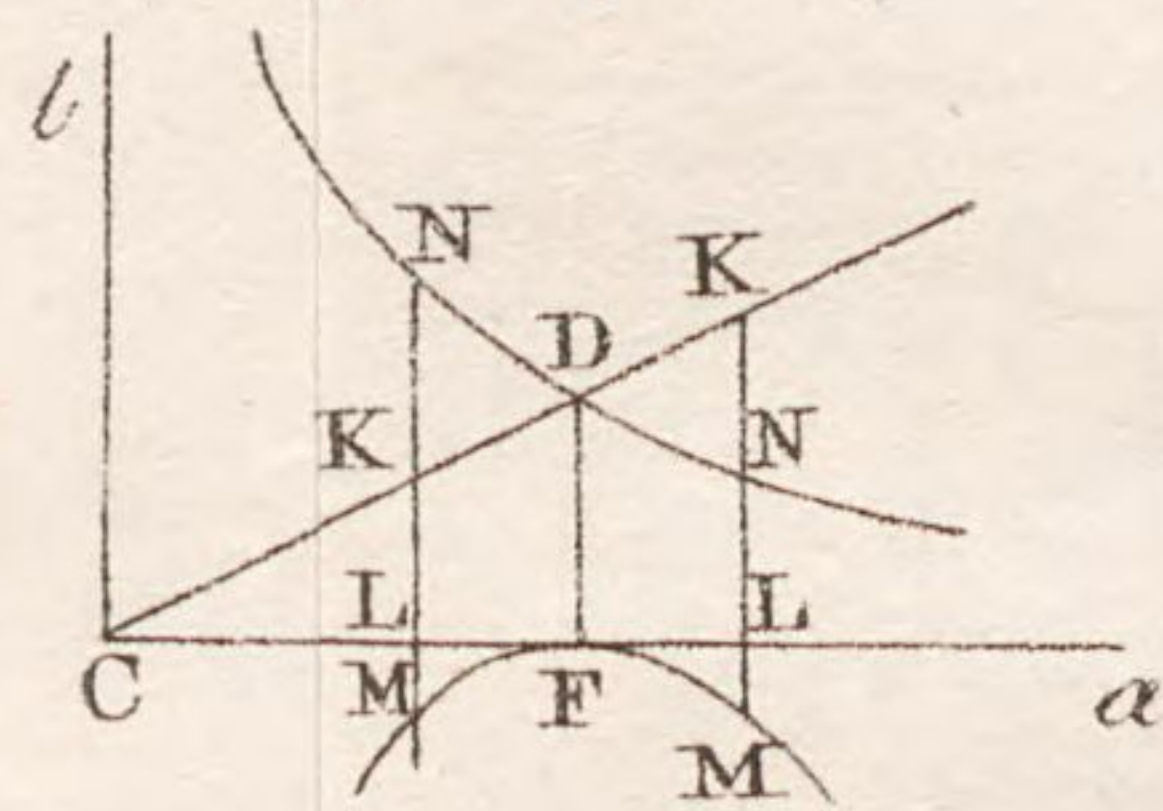


Fig. 345.

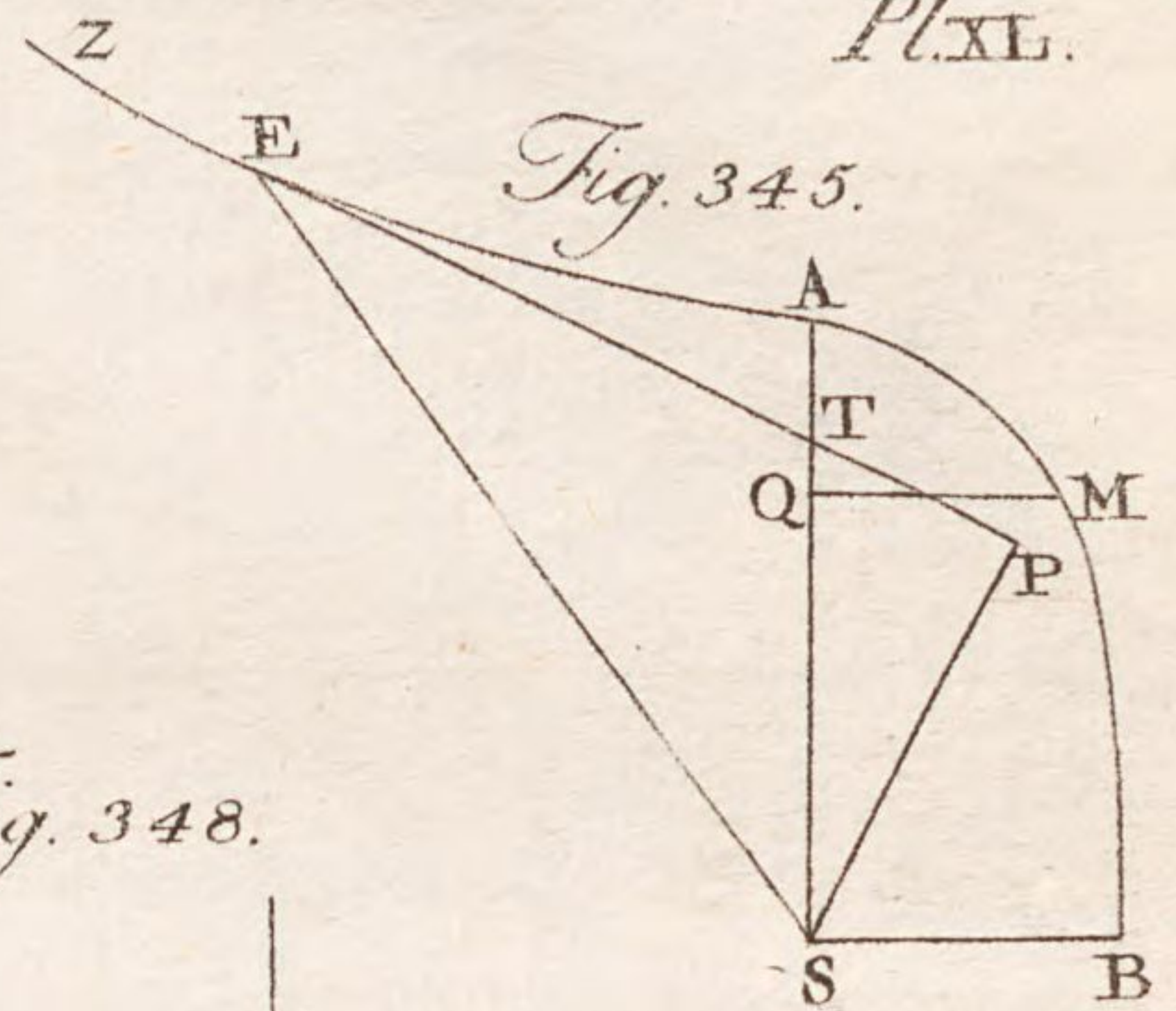


Fig. 346.

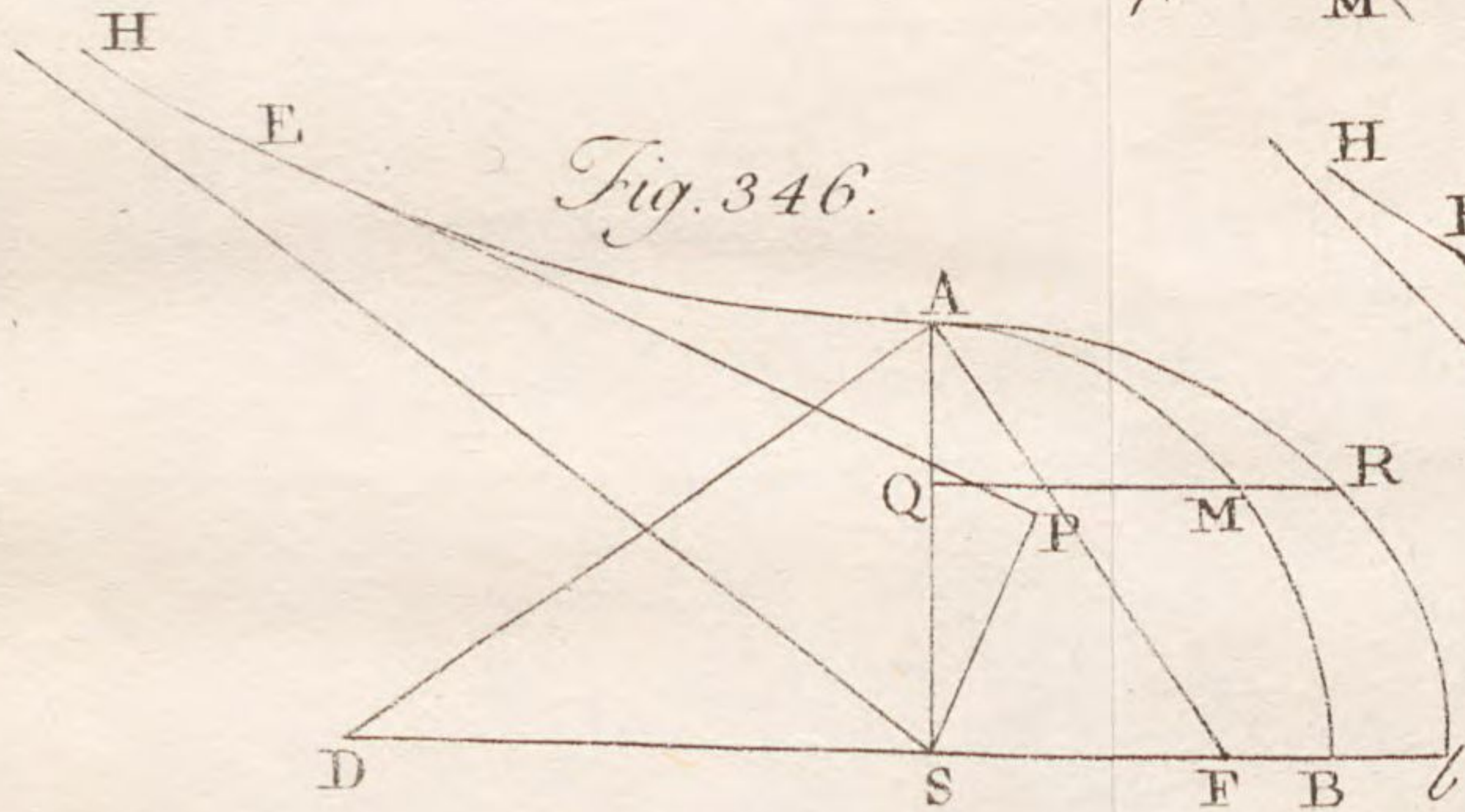


Fig. 348.

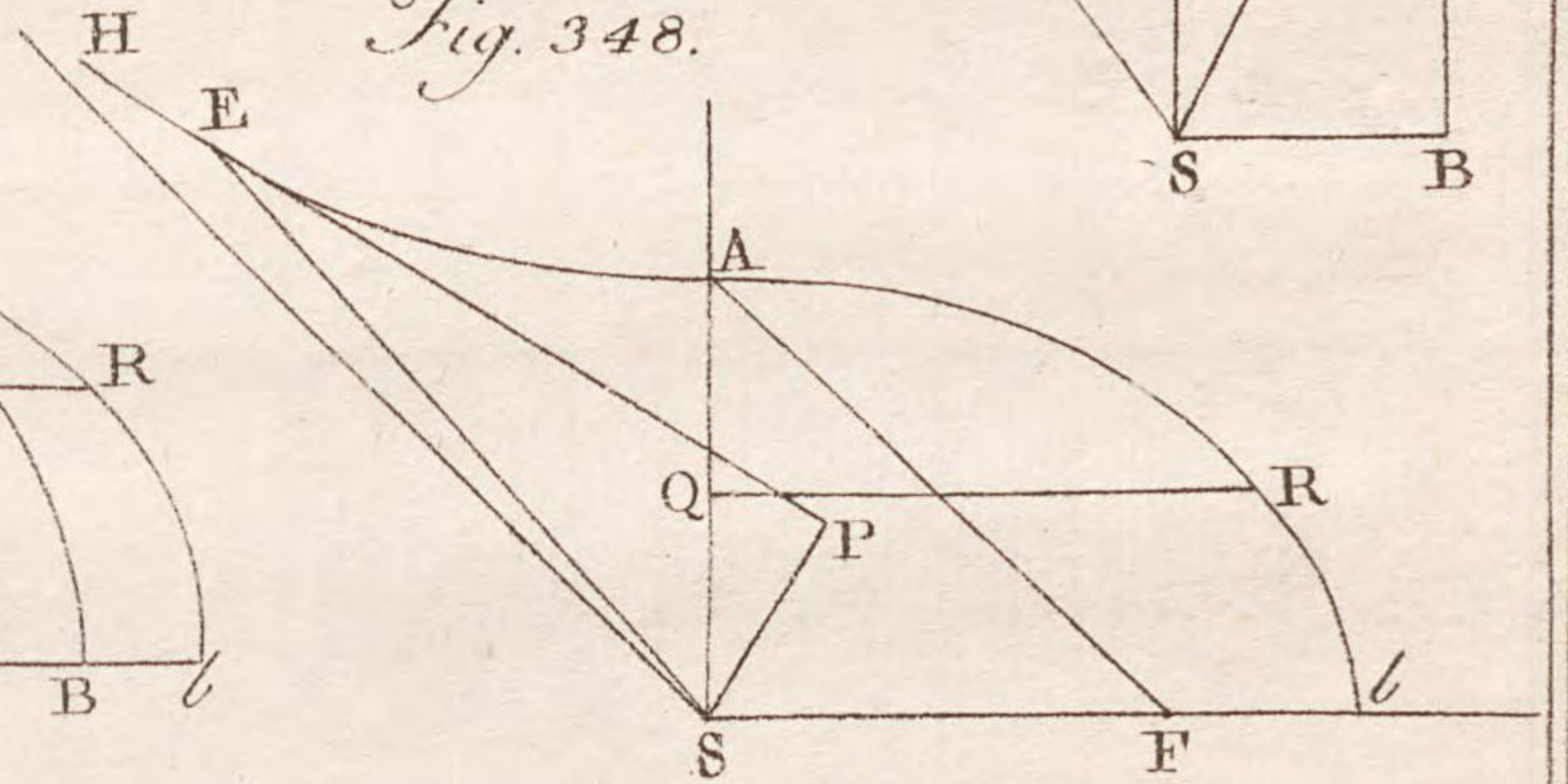


Fig. 347.

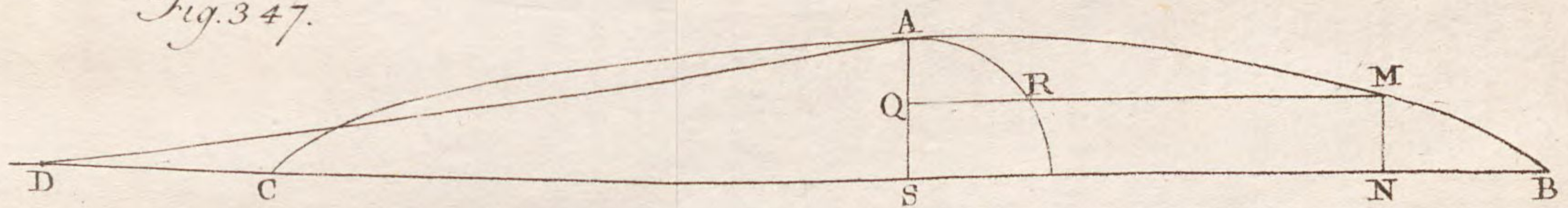


Fig. 349.

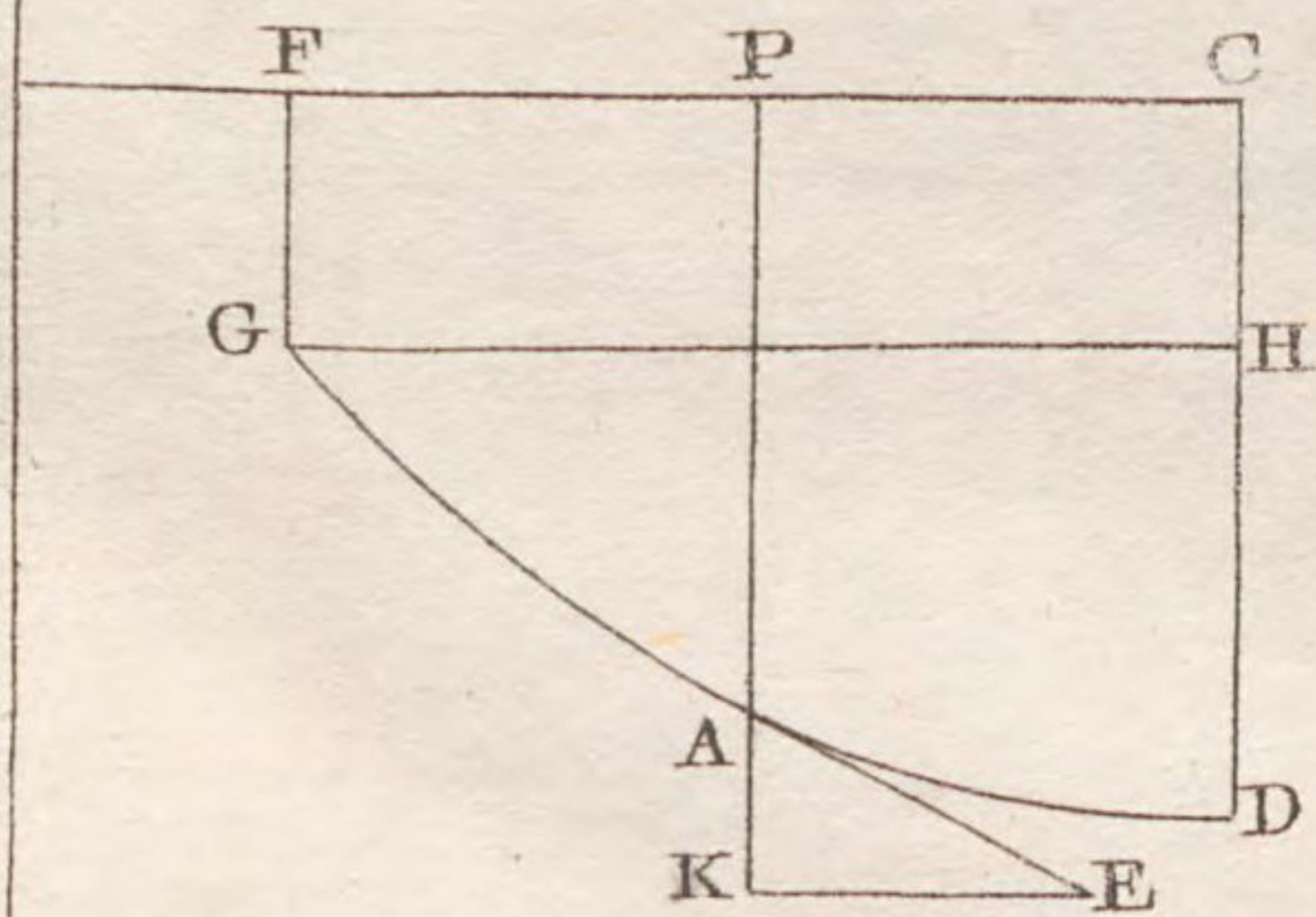


Fig. 350.

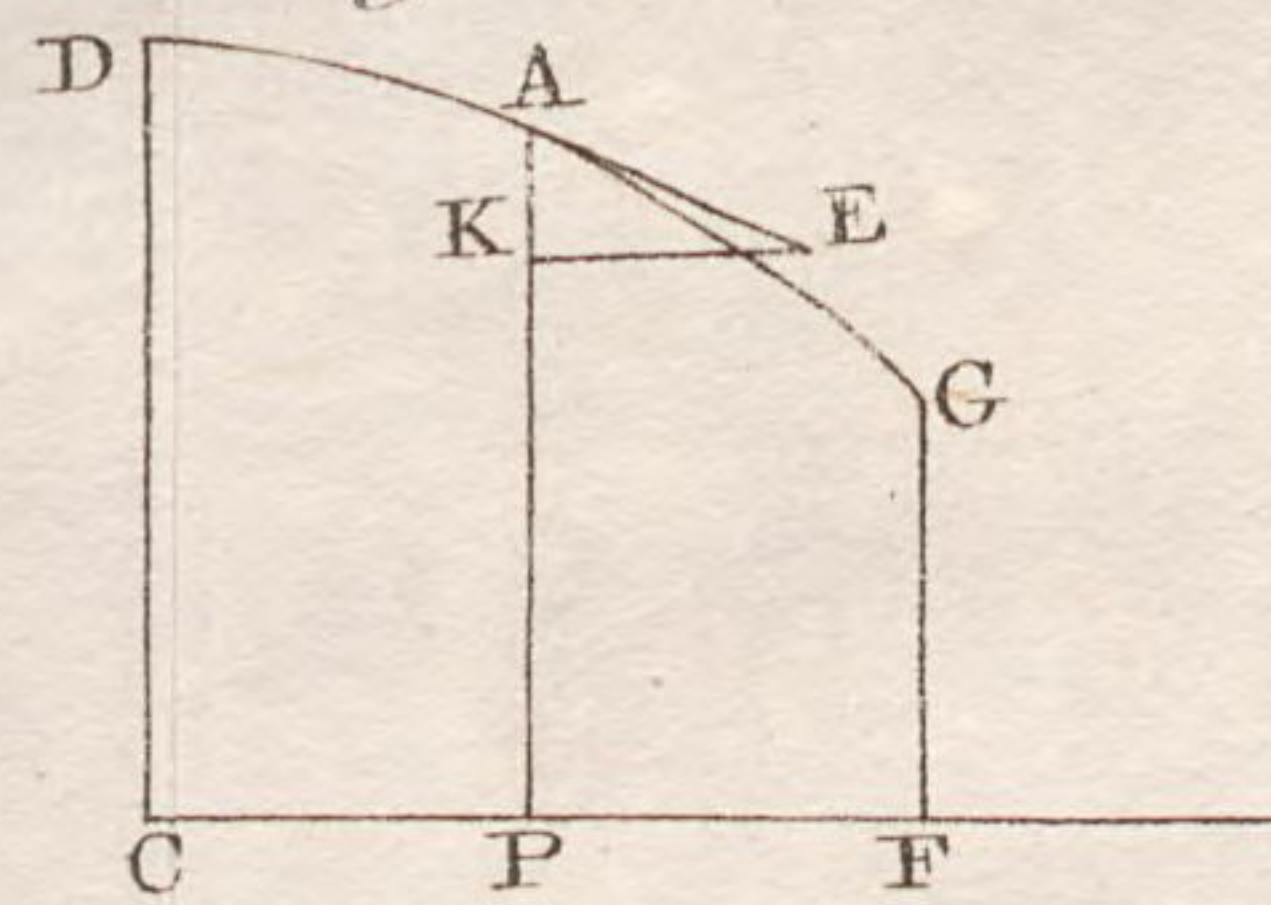


Fig. 351.

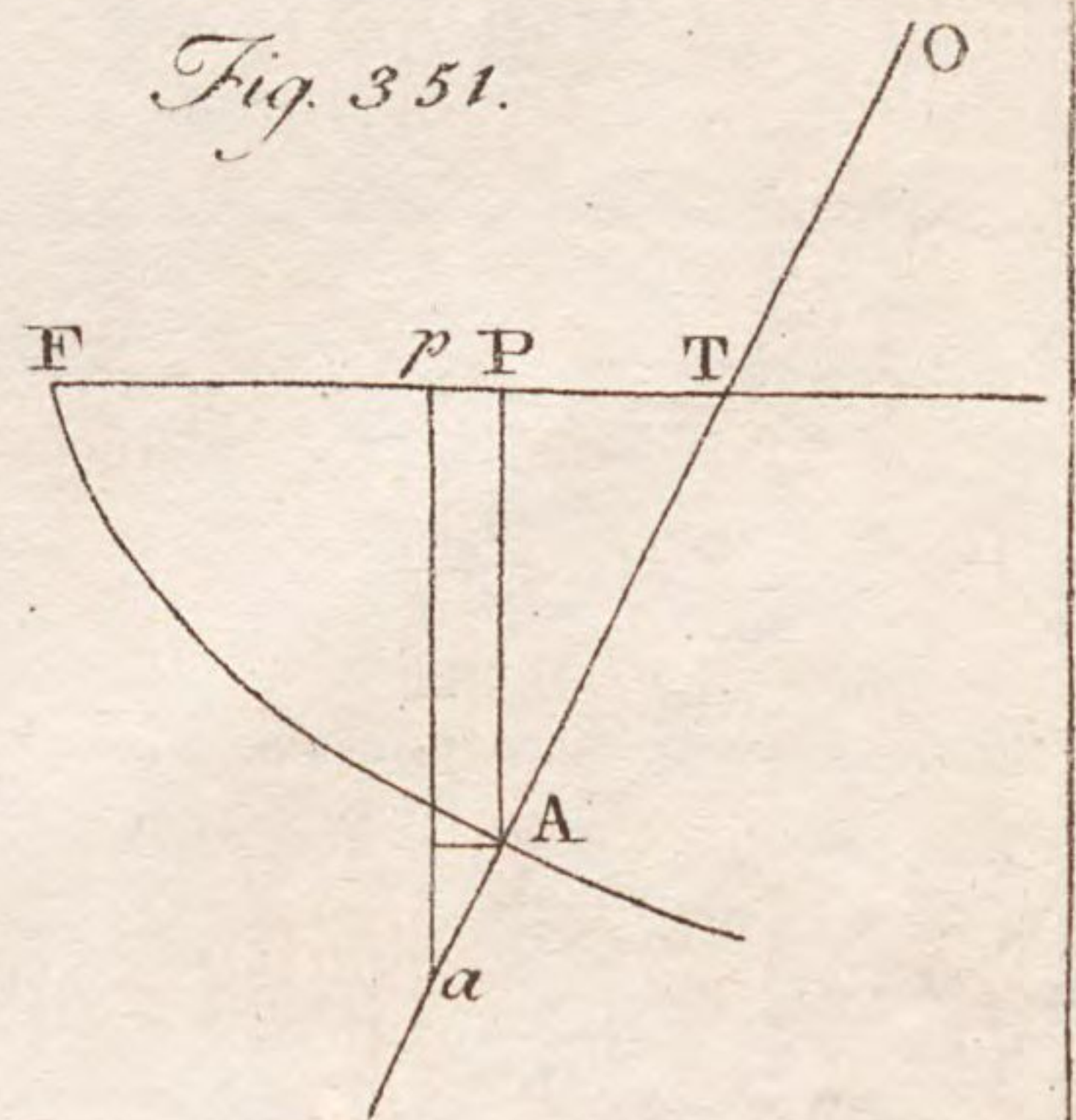


Fig. 352.

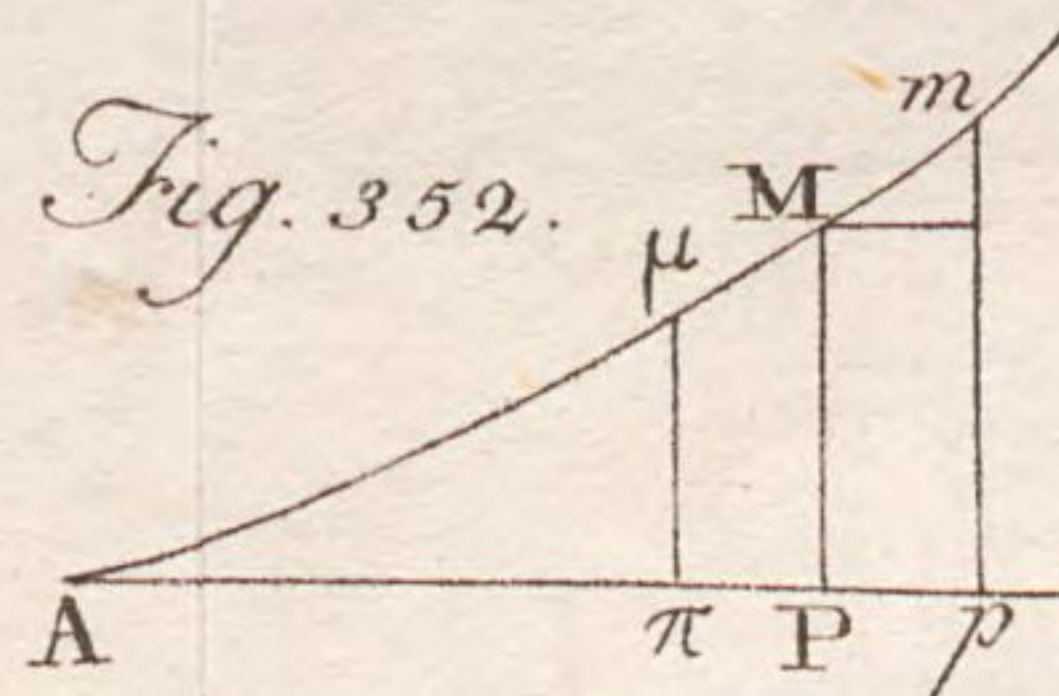




Fig. 149.

A T A B L E OF THE Principal Contents.

INTRODUCTION.

T HE design of this Treatise,	Page 1
Of the method of exhaustions, from the 12th book of the Elements,	4
Elliptic and circular areas compared by this method,	7
A general theorem concerning figures described about a conic section, or inscribed in it,	8
Propositions from Archimedes concerning spheres, spheroids, &c.	9
A general property of the solids that is generated by a conic section revolving about its axis,	24
The quadrature of the parabola after Archimedes,	27
Of the spiral of Archimedes,	30
The quadrature of a spiral by Pappus,	31
Remarks on the method of the Ancients,	33
On the methods of indivisibles and infinitesimals,	37

BOOK I.

C HAPTER I. Of the grounds of the method of fluxions.	
Definitions and illustrations,	Article 1
The axioms,	15
Theorems concerning uniform motions from Archimedes,	16
C c c c c 2	Theorems

<i>Theorems concerning variable motions,</i>	18
<i>Of comparing the fluxions of quantities by determining the limit of the ratio of their increments or decrements,</i>	66
<i>Of second fluxions,</i>	70
CHAP. II. <i>Of the fluxions of plane rectilineal figures.</i>	
<i>Of the fluxion of a parallelogram of an invariable altitude, Art.</i>	78
<i>Of the fluxion of a triangle,</i>	81
<i>The increment of the triangle resolved into two parts, that which measures the generating motion, and that which measures its acceleration,</i>	93
<i>The theory of motions that are accelerated or retarded uniformly,</i>	94
<i>Of the fluxion of a rectangle,</i>	98
CHAP. III. <i>Of the fluxions of plane curvilineal figures.</i>	
<i>Of the fluxion of an area, the ordinates being supposed parallel, Art.</i>	105
<i>General corollaries relating to the theory of motion,</i>	114
<i>Of the fluxion of the area generated by a ray revolving about a given center,</i>	116
<i>Of similar curvilineal figures,</i>	122
CHAP. IV. <i>Of the fluxions of solids, Art.</i>	124
<i>Illustrations of second and third fluxions,</i>	128
CHAP. V. <i>Of the fluxions of quantities that are in a continued geometrical progression, the first term of which is invariable, Art.</i>	140
CHAP. VI. <i>Of logarithms, and the fluxions of logarithmic quantities.</i>	
<i>An account of logarithms from Napier the inventor,</i>	151
<i>Of the fluxions of quantities that increase or decrease proportionally,</i>	158
<i>Of the fluxions of quantities, when their logarithms are in an invariable ratio,</i>	165
<i>Of the fluxions of quantities that are represented by powers with irrational or variable exponents,</i>	168
<i>Of the second, third, and higher fluxions of a quantity that increases proportionally,</i>	169
<i>Theorems for approximating to the value of logarithms,</i>	171
<i>Of the different logarithmic systems and the ratio modularis, Of</i>	174

The C O N T E N T S.

757

<i>Of the logarithmic curve,</i>	176
<i>Of hyperbolic areas,</i>	177
<i>Of the analogy betwixt circular arks and logarithms,</i>	178
CHAP. VII. Of tangents.	
<i>Definitions,</i>	Art. 180
<i>Of the fluxion of the base, ordinate and curve,</i>	184
<i>Of the fluxion of the ark, sine, tangent, secant, &c.</i>	192
<i>Of the fluxions of the curve, the ray drawn to the curve from a given point, and the circular ark described from that point as center,</i>	199
<i>Of the fluxions of angles,</i>	208
<i>Theorems concerning tangents,</i>	211
CHAP. VIII. Of the fluxions of curve surfaces.	
<i>Lemmas concerning conical surfaces,</i>	216
<i>Of the fluxion of a curve surface,</i>	228
<i>Of the surfaces generated by a circular arch about any chord,</i>	231
<i>Of the surfaces generated by any arks, the center of gravity of an arch, and the theorem of Guldinus,</i>	233
CHAP. IX. Of the usual rule for determining the greatest and least ordinates,	
<i>Of the analogy betwixt the inverse method of tangents, and the quadrature of figures,</i>	247
<i>A more accurate rule for finding the greatest and least ordinates,</i>	261
<i>A similar rule for finding the points of contrary flexure,</i>	263
<i>Of cuspids of various kinds,</i>	268
<i>Of the greatest and least rays that can be drawn from a given point to a curve,</i>	277
<i>Other rules for finding the points of contrary flexure and cuspids,</i>	279
CHAP. X. Of the asymptotes of curve lines, &c.	
<i>Definition of asymptotes, with examples,</i>	Art. 286
<i>Of the parts of geometrical magnitude,</i>	290
<i>Of asymptotes, and the areas bounded by them and the curves,</i>	292
<i>Of the solid generated by this area,</i>	307
<i>Examples of constructions for determining the tangents and asymptotes of curves that are described by the revolution of lines or angles,</i>	313
<i>Theorems for discovering whether a figure hath an asymptote and</i>	

<i>and the area bounded by it and the curve hath an assignable limit which it cannot exceed,</i>	325
<i>Of the surface generated by the curve about the asymptote,</i>	339
<i>Of spiral lines and their areas,</i>	340
<i>Of the limits to which the sums of progressions approach, with examples and theorems for approximating to those limits,</i>	350
CHAP. XI. <i>Of the curvature of lines, &c.</i>	
<i>Definitions,</i>	Art. 363
<i>Theorems for finding the curvature and its variation in geometrical figures, and for comparing the different degrees of contact of the curve and circle of curvature,</i>	365
<i>Examples in the conic sections,</i>	371
<i>Of the curvature that is less than that of any circle,</i>	377
<i>Of the curvature that is greater than in any circle,</i>	378
<i>Other theorems concerning the curvature and its variation,</i>	381
<i>A general property of the lines of the third order, when two tangents can be drawn to the line from a point in it,</i>	401
<i>Of the evolution of lines,</i>	402
<i>Of the properties of the cycloid, and the descent of a heavy body along it.</i>	
<i>Of the caustics by reflexion,</i>	409
<i>Caustics by refraction,</i>	413
<i>Of the rays that define the first and second rainbow,</i>	415
<i>Of centripetal forces,</i>	416
<i>The ratio of the velocity in a curve to the velocity in a circle at the same distance from the center in a void or medium.</i>	424
<i>The construction of the trajectory, when the velocity is such as would be acquired by an infinite descent,</i>	436
<i>Of motions in a conic section,</i>	445
<i>The cases distinguished wherein a body may revolve betwixt the higher and lower apsides, and when it continually approaches to the center or recedes from it,</i>	447
<i>Of the resistance and density of the medium in which a given trajectory is described,</i>	452
<i>Of gravitation towards several centers,</i>	462
<i>Of the motion of the nodes of the moon,</i>	480
<i>Of the variation of the inclination of the plane of the lunar orbit,</i>	487
	Of

<i>Of the acceleration of the area described by the moon about the earth,</i>	490
<i>Of fluids that gravitate towards several centers,</i>	491
<i>Of the figure of a fluid that gravitates towards a center and revolves about an axis,</i>	492
<i>Of the intersection of the curve and circle of curvature,</i>	493

VOLUME II.

C HAP. XII. <i>Of the method of infinitesimals, of the limits of ratios, and of the general theorems which are derived from this doctrine, for the resolution of geometrical and philosophical problems.</i>	
<i>Of the harmony betwixt the method of fluxions and of infinitesimals,</i>	495
<i>Some objections against the method of infinitesimals considered,</i>	498
<i>The true reason why parts of the element are to be neglected in the method of infinitesimals,</i>	501
<i>Of Sir Isaac Newton's method by the limits of ratios,</i>	502
<i>Propositions of the preceeding chapters demonstrated briefly by this method,</i>	506
<i>Theorems concerning the center of gravity and its motion, and their use shown in resolving several problems concerning the collisions of bodies,</i>	510
<i>Of the descent of bodies that act upon one another, of the descent and ascent of their center of gravity, and the preservation of the vis ascendens or vis viva,</i>	521
<i>Of the center of oscillation,</i>	534
<i>Of the motion of water issuing from a cylindric vessel,</i>	537
<i>Of the motion of water issuing from any vessel,</i>	550
<i>Of the Catenaria, when gravity acts in parallel lines,</i>	551
<i>General theorems concerning the trajectories, lines of swiftest descent, the Catenaria, &c,</i>	563
C HAP. XIII. <i>The analysis of the problem concerning the lines of swiftest descent, when an uniform or variable gravity acts in parallel lines,</i>	572
<i>The synthetic demonstration,</i>	576
	<i>The</i>

<i>The same, when gravity tends to a given center,</i>	578
<i>Another synthetic demonstration,</i>	584
<i>Of the lines of swiftest descent amongst those of the same perimeter in any hypothesis of gravity,</i>	588
<i>The first general isoperimetrical problem resolved by first fluxions, and the resolution demonstrated synthetically,</i>	592
<i>The problem extended further by the same method,</i>	597
<i>The second general isoperimetrical problem resolved in the same manner,</i>	601
<i>The property of the solid of least resistance demonstrated in this manner,</i>	606
CHAP. XIV. <i>Of the ellipse considered as the section of a cylinder,</i>	609
<i>General properties of the conic sections transferred briefly from the circle,</i>	622
<i>Of gravitation towards spheres and spheroids,</i>	628
<i>Supposing the density of the planets uniform, their figure is accurately that of the oblate spheroid, which is generated by the conic ellipse about its second axis,</i>	636
<i>Of the figure of the planets and variation of gravity towards them,</i>	641
<i>The gravitation at the pole and equator, or any point on the surface of a spheroid, measured accurately by circular arcs or logarithms,</i>	642
<i>The gravitation in the axis or plane of the equator produced, measured accurately by the same,</i>	648
<i>Of the figure of the earth in particular, supposing its density uniform,</i>	655
<i>Of the gravity towards a spheroid, supposing the density variable,</i>	660
<i>Of the figure of Jupiter, and the effects of his spheroidical form upon the motions of the Satellites,</i>	682
<i>Of the Tides,</i>	686
<i>Of other laws of attraction,</i>	696

BOOK II.

Of the computations in the method of fluxions.

CHAP. I. <i>Of the fluxions of quantities considered abstractly as represented by general characters in Algebra.</i>	
<i>Of the import of some algebraic symbols,</i>	Art. 6927
<i>The principles of this method adapted to algebra,</i>	700
<i>Of the fluxions of powers of all kinds,</i>	707
<i>Of the fluxions of products and quotients,</i>	715
<i>Of the fluxions of logarithms,</i>	717
<i>Of second and higher fluxions,</i>	720
CHAP. II. <i>Of the notation of fluxions,</i> Art. 723	
<i>The rules of the direct method,</i>	724
<i>The fundamental rules of the inverse method,</i>	735
<i>Of infinite series,</i>	745
<i>An investigation of the binomial and multinomial theorems,</i>	748
<i>Other theorems,</i>	751
<i>Examples of their use,</i>	753
CHAP. III. <i>Of the analogy betwixt elliptic and hyperbolic sectors,</i> 758	
<i>Of resolving trinomials into quadratic divisors,</i>	765
<i>Of reducing fluents to circular arks and logarithms, when the fluxion is expressed by rational quantities,</i>	770
<i>Of reducing fluents to the same measures, when the fluxion involves an irrational binomial or trinomial,</i>	789
<i>Of reducing fluents to hyperbolic and elliptic arks,</i>	798
<i>Of reducing fluents of a higher kind to others of a more simple form,</i>	810
CHAP. IV. <i>Of the area when the ordinate is expressed by a fluent,</i> Art. 813	
<i>Of the area, when the ordinate and base are both expressed by fluents,</i>	819
<i>Instances wherein the total area or fluent is measured by circular arks or logarithms, when it does not appear that the same fluent can be generally reduced to those measures,</i>	822
<i>Theorems derived from the method of fluxions for approximating to the sums of progressions by areas, and conversely,</i>	828
<i>Theorems for finding the sum of any powers, positive or negative,</i>	
D d d d d	of

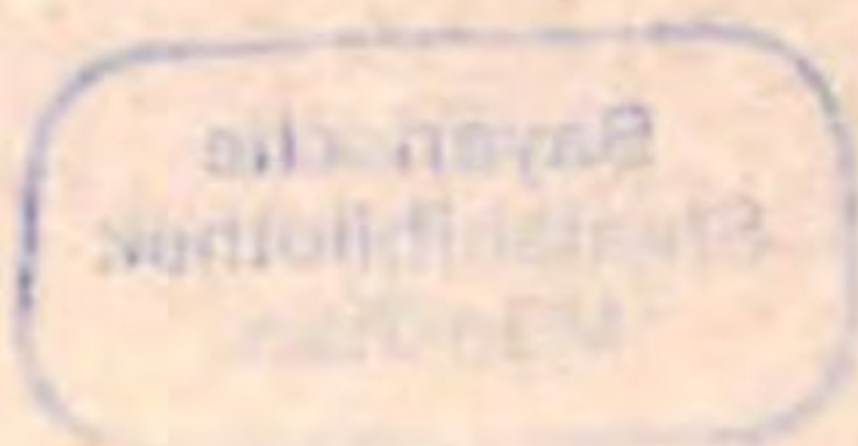
<i>of the terms in an arithmetical progression, and for finding the sums of their logarithms,</i>	833
<i>Of the ratio of the sum of all the uncix of a binomial of a very high power to the uncia of the middle term,</i>	844
<i>Of computing the area from a few equidistant ordinates,</i>	848
<i>Theorems derived from the method of fluxions, for interpolating the intermediate terms of a series,</i>	850
CHAP. V. <i>Of the general rules for the resolution of problems by computations, with examples.</i>	
<i>Of the rules for determining the tangents,</i>	Art. 857
<i>The greatest and least ordinates,</i>	858
<i>The points of contrary flexure and cuspids,</i>	866
<i>The center of curvature,</i>	870
<i>The caustics by reflexion and refraction,</i>	872
<i>The centripetal forces.</i>	874
<i>The construction of the trajectory that is described by a force, which is inversely as the fifth power of the distance, by logarithms in certain cases,</i>	878
<i>In these cases a body may recede from the center continually, so as never to arise to a certain altitude, or may approach to it forever and never descend to a certain distance,</i>	879
<i>The construction in other cases,</i>	881
<i>The rules for computing the time of descent along a given curve,</i>	884
<i>The Time in a finite circular arch measured by the arcs of conic sections,</i>	886
<i>The same by infinite series,</i>	887
<i>Rules concerning the computation of motions in a medium,</i>	888
<i>Rules for determining the figure of the catenaria, and the lines of swiftest descent,</i>	889
<i>Rules for the computation of areas, solids, curvilineal arcs and surfaces,</i>	890
<i>The meridional parts in a spheroid computed by circular arcs or logarithms,</i>	895
<i>The gravitation towards a spheroid at the Pole and Equator, measured by circular arcs or logarithms, when the force towards any particle is inversely as any power of the distance from it,</i>	900
<i>Of the centers of gravity and oscillation,</i>	906
	Of

The CONTENTS.

763

<i>Of the proportion of the power to the weight, that a machine may have the greatest effect,</i>	907
<i>Of the same, when the friction is considered,</i>	908
<i>The most advantageous position of a plane, which moves parallel to itself with a given direction, that a stream may impell it with the greatest force, when the velocities of the stream and plane are given,</i>	910
<i>The wind ought to strike the sails of a Wind-mill in a greater angle than $54^{\circ} 44'$,</i>	914
<i>The most advantageous position of the sails, that the wind may impell a ship with the greatest force in a given direction, the velocities of the wind and ship being given,</i>	916
<i>How an ark is to be divided into any number of parts, that the product of any powers of the sines of the several parts may be a maximum,</i>	921
<i>The most advantageous direction of the motion of a ship, and best position of the sail, that the ship may recede from a given line or coast with the greatest velocity,</i>	922
<i>Of reducing equations from second to first fluxions, with examples,</i>	924
<i>The construction of the elastic curve, and of other figures, by the rectification of the conic sections,</i>	927
<i>Of the vibrations of musical chords,</i>	929
<i>Problems concerning the maxima and minima that are proposed with limitations concerning the perimeter of the figure, its area, the solid generated by this area, &c. resolved by first fluxions.</i>	931
<i>Examples of this kind relating to the solid of least resistance,</i>	934
<i>An example of the method of computing from the general principles in art. 563.</i>	935
<i>An instance of the theorems by which the value of the ordinate may be determined from the value of the area, by common Algebra,</i>	936
<i>It is relative not absolute space and motion that are supposed in the method of fluxions.</i>	937

ERRATA.



ERRATA.

- P**Age 17. line 8. for CR read GR.
 21. l. 2. for B r. R.
 26. l. 18. r. $\frac{1}{4}$ of a similar cone.
 111. l. 9. for AM becomes, r. AP becomes.
 119. l. 6. from the bottom of the page, for P. r. p.
 128. l. 26. for AC in Y, r. MY in S.
 129. l. 1. for ET r. FT.
 153. l. 29. for rn r. rn.
 179. l. 21. after tangent, add, and in some cases on the same side of it.
 181. l. 22. for fig. 46. r. fig. 47.
 197. l. 18. for decreases r. increases.
 197. l. 12. for ASE r. ACE
 200 l. 9. and 13. for V. r. O, and for Q r. Q.
 202 l. 35. for mh r. mg, & f. Bb r Bg.
 203 l. 10. and 16. for u r. v.
 204. l. 7. for CT less than TB r. TB less than CT.
 215. l. 18. for three r. four.
 219. l. 9. for in E r. in T.
 223. l. 31. for eUN r. euN.
 251. l. 12. after further, add, in this Place, but see art. 331.
 279. l. 24. dele N. 2.
 311. l. 28. after Emh. r. having its axis parallel to EB.
 317. l. 12. r. passes thro' the focus.
 331. l. 4. from the bottom, for AM r. Am, and l. 3. r tangent of the angle.
 332. l. 9. from the bottom for Pa, r. CP.
 339. l. 8. for to AP r. to the fluxion of AP.
 345. l. 23. dele and I.
 355. l. 26. and 35. after difference r. or sum.
 357. l. 15. for $m-1$ r. $n-1$.
 370. l. 15. for XZ r. EZ, and l. 16. for 8th r 6th.
 383. l. ult. for as Aa is, r. in the subduplicate ratio of Aa.
 390. l. 14. for Dl r. Dh, and for Dh r. Dl.
 401. l. 3. from the bottom of the page, for Rr. r. Pr.
 404. l. 25. for 180. r. 480.
 408. l. 18. for 204, r. 205, and l. 20. for $\sqrt{G-2V}$ r. $G-2V$.
 410. l. 11. for $\frac{1}{2}L$ r. 2L. and l. 25. for next r. 14th.
 426. l. 8. for CL. r. CP.
 490. l. 18. for TB. r. TR.
 496. l. 34. for Bb r. B.
 497. l. ult. for lmp r. lMp.
 498. l. 20. for pml r. pMl.
 502. l. 3. for FEa r. FEQ.
 505 l. 28. for AED r. AEQ. and l. ult. for EQ r. Eq.
 507. l. 10. for SK r. SX.
 516. l. 20. 21. and 22. for L. r R. and for l. r. r.
 518. l. 18. and 29. for KL r. Kl, and l. 23. for CK $\times$ CQ r. CG $\times$ KQ.
 519. l. 23. dele and.
 520. l. 28. for of them. r. of the right lines.
 556. l. 10. for z^2 r. z^3 , and for b^2 r. b^3 .
 601. l. 1. for $m+1$ r. $m+e$.
 613. l. 14. for CM : CP, r. CP : CM.
 616. l. 13. for AH r. OH, and l. 15. for HI r. HO.
 618 l. 12. for Qq r. 2Qq.
 620. l. 11. for $\sqrt{aa-xx}$ r. $\sqrt{xx-aa}$.
 623 l. 2. for ax r. -ax, l. 3. for az r. -az, and l. 5. for $\frac{ap}{q\sqrt{-1}}$ r. $\frac{aq}{q\sqrt{-1}}$.
 625. l. 30. for FIG. 300. r. FIG. 302.
 627. l. 5. for 22 r. 2z.
 639. l. 8. for 301. r. 302.
 655. l. 18. for An r. Am.
 658. l. 3. for f r. k.
 660. l. 15. for 9×6 r. 6×8 .
 673. l. 23. for CK r. CD.
 691. l. 8. for Pr' r. PT'
 709. l. 15. for EP r. Ep.
 712. l. 17. for CS r. $\frac{1}{2}$ CS.
 720 l. 5. for CE r. CH.
 721. l. 1. under b^3cu , for b^3 r b^4 .
 745. l. ult. and 746. l. 11. for ACr. BC.

